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GEOMETRICAL EXERCISES

IN

PAPER FOLDING.

BY

T. SUNDARA ROW, B.A.,
Deputy Collector.

Madras:
Printed by ADDISON & CO., Mount Road.

1893.

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TO

THE HON. SIR HENRY STOKES, B.A., K.C.S.I.,

THIS BOOK IS DEDICATED

BY HIS

KIND PERMISSION.

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INTRODUCTION.

THE *idea* of this book was suggested to me by Kindergarten Gift No. VIII.—Paper-folding. The gift consists of 200 variously coloured squares of paper, a folder, and diagrams and instructions for folding. The paper is coloured and glazed on one side. The paper may, however, be of self-colour, alike on both sides. In fact, any paper of moderate thickness will answer the purpose, but coloured paper shows the creases better, and is more attractive. The kindergarten gift is sold by Messrs. Higginbotham and Co.; but coloured paper of both sorts can be had in the bazaars. A packet of 100 squares of both sorts accompanies this book, and the packets can also be had separately. Any sheet of paper can be cut into a square as explained in the opening articles of this book, but it is neat and convenient to have the squares ready cut.

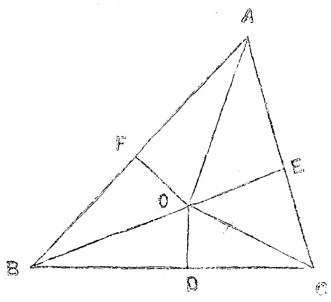
2. These exercises do not require mathematical instruments, the only things necessary being a penknife and scraps of paper, the latter being used for setting off equal lengths. The squares are themselves simple substitutes for a straight edge and a T square.

3. In paper-folding several important geometrical processes can be effected much more easily than with a pair of compasses and ruler, the only instruments the use of which is sanctioned in Euclidian Geometry; for example, to divide straight lines and angles into two or more equal parts, to draw perpendiculars and parallels to straight lines. It is, however, not possible in paper-folding to describe a circle, but a number of points on a circle, as well as other curves, may be obtained by other methods. These exercises do not consist merely of drawing geo

metrical figures involving straight lines in the ordinary way, and folding upon them, but they require an intelligent application of the simple processes peculiarly adapted to paper-folding. This will be apparent at the very commencement of this book.

4. The use of the kindergarten gifts not only affords interesting occupations to boys and girls, but also prepares their minds for the appreciation of science and art. Conversely the teaching of science and art later on can be made interesting and based upon proper foundations by reference to kindergarten occupations. This is particularly the case with Geometry, which forms the basis of every science and art. The teaching of Euclid in schools can be made very interesting by the free use of the kindergarten gifts. It would be perfectly legitimate to require pupils to fold the diagrams on paper. This would give them neat and accurate figures, and impress the truth of the propositions forcibly on their minds. It would not be necessary to take any statement on trust. But what is now realised by the imagination and idealization of clumsy figures can be seen in the concrete. A fallacy like the following would be impossible.

5. *To prove that every triangle is isosceles.* Let ABC be any triangle. Bisect BC in D, and through D draw DO perpendicular to BC. Bisect the angle BAC by AO.



(1) If AO and DO do not meet, they are parallel. Therefore AO is at right angles to BC. Therefore AB = AC.

(2) If AO and DO do meet, let them meet in O. Draw OE perpendicular to AC and OF perpendicular to AB. Join OB, OC. By Euclid I. 26 the triangles AOF and AOE are equal; also by Euclid I. 47 and I. 8 the triangles BOF and COE are equal. Therefore

$$AF+FB=AE+EC,$$

$$i.e.\ AB=AC.$$

It will be seen by paper-folding that, whatever triangle be taken, AO and DO cannot meet within the triangle.

O is the midpoint of the arc BOC of the circle which circumscribes the triangle ABC.

6. Paper-folding is not quite foreign to us. Folding paper squares into natural objects—a boat, double boat, ink bottle, cup-plate, &c., is well known, as also the cutting of paper in symmetrical forms for purposes of decoration. In writing Sanskrit and Mahrati, the paper is folded vertically or horizontally to keep the lines and columns straight. In fair copying letters in public offices an even margin is secured by folding the paper vertically. Rectangular pieces of paper folded double have generally been used for writing, and before the introduction of machine cut letter paper and envelopes of various sizes, sheets of convenient size were cut by folding and pulling asunder larger sheets, and the second half of the paper was folded into an envelope enclosing the first half. This latter process saved paper and had the obvious advantage of securing the post marks on the paper written upon. Paper-folding has been resorted to in teaching the XIth Book of Euclid, which deals with figures of three dimensions. But it has seldom been used in respect of plane figures. Mr. B. Hanumanta Row, B.A., has done it. In his First Lessons in Geometry, he has made frequent

allusions to it, but the hint has not been generally taken by teachers.

7. I have attempted not to write a complete treatise or text-book on Geometry, but to show how regular polygons, circles and other curves can be folded or pricked on paper. I have taken the opportunity to introduce to the reader some well known problems of ancient and modern Geometry, and to show how Algebra and Trigonometry may be advantageously applied to Geometry, so as to elucidate each of the subjects which are usually kept in separate pigeonholes.

8. The first nine chapters deal with the folding of the regular polygons treated in the first four books of Euclid, and of the nonagon. The paper square of the kindergarten has been taken as the foundation, and the other regular polygons have been worked out thereon. Chapter I. shows how the fundamental square is to be cut and how it can be folded into equal right-angled isosceles triangles and squares. Chapter II. deals with the equilateral triangle described on one of the sides of the square. Chapter III. is devoted to the Pythagorean theorem (Euclid I. 47) and the propositions of the second book of Euclid and certain puzzles connected therewith. It is also shown how a right-angled triangle with a given altitude can be described on a given base. This is tantamount to finding points on a circle with a given diameter.

9. Chapter X. deals with the Arithmetical, Geometrical, and Harmonic progressions and the summation of certain arithmetical series. In treating of the progressions, lines whose *lengths* form a progressive series are obtained. A rectangular piece of paper chequered into squares exemplifies A.P. For the G.P. the properties of the right-

angled triangle, that the altitude from the right-angle is a mean proportional between the segments of the hypotenuse, and that either side is a mean proportional between its projection on the hypotenuse and the hypotenuse, are made use of. In this connection the Delian problem of duplicating a cube has been explained. In treating of Harmonic progression, the fact that the bisectors of an interior and corresponding exterior angle of a triangle divide the opposite side in the ratio of the other sides of the triangle has been used. This affords an interesting method of graphically explaining *systems in involution.* The sums of the natural numbers and of their cubes have been obtained graphically, and the sums of certain other series have been deduced therefrom.

10. Chapter XI. deals with the general theory of regular polygons, and the calculation of the numerical value of π. The propositions in this chapter are very interesting.

11. Chapter XII. explains certain general principles, which have been made use of in the preceding chapters,—Congruency, Symmetry and Similarity of figures, Concurrency of straight lines, and Collinearity of points are touched upon.

12. Chapters XIII. and XIV. deal with the Conic Sections and other interesting curves. As regards the circle, its harmonic properties among others are treated. The theories of *inversion* and *co-axal circles* are also explained. As regards other curves it is shown how they can be marked on paper by paper-folding. The history of some of the curves is given, and it is shown how they were utilized in the solution of the classical problems, to find two geometrical means between two given lines, and to trisect

a given rectilineal angle. Although the investigation of the properties of the curves involves a knowledge of advanced mathematics, their *genesis* is easily understood and is interesting.

13. I have sought not only to aid the teaching of Geometry in schools and colleges, but also to afford mathematical recreation to young and old, in an attractive and cheap form. "Old boys" like myself may find the book useful to revive their old lessons, and to have a peep into modern developments which, although very interesting and instructive, have been ignored by the Madras University.

GEOMETRICAL EXERCISES

IN

PAPER FOLDING.

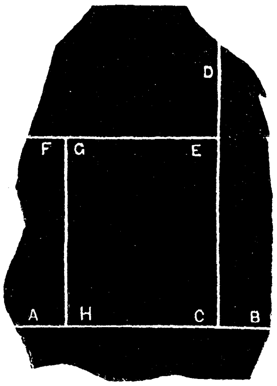
CHAPTER I.

THE SQUARE.

THE upper side of a piece of paper lying flat upon a table is a plane surface, and so is the lower side which is in contact with the table.

2. The two surfaces are separated by the material of the paper. The material being very thin, the other sides of the paper do not present appreciably broad surfaces, and the edges of the paper are practically lines. The two surfaces though distinct are inseparable from each other.

3. Look at this irregularly shaped piece of paper, and at this piece of letter paper which is rectangular. Let us try and shape the former paper like the latter.



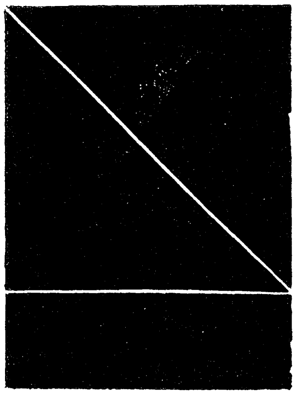
4. Place the irregularly shaped piece of paper upon the table, and fold it flat upon itself. Let AB be the crease thus formed. It is straight. Now pass a knife along the fold and separate the smaller piece. We thus obtain one straight edge.

5. Fold the paper again as before along CD, so that the edge AB is doubled upon itself. Unfolding the paper, we see that the crease CD is at right angles to the edge AB. It is evident by superposition that the angle ACD = the angle BCD, and that each of these angles = an angle of the letter paper. Now pass a knife as before along the second fold, and remove the smaller piece.

6. Repeat the above process, and obtain the edges EF and GH. It is evident by superposition that the angles at C, E, G and H are right angles, equal to each other, and that the sides CE, EG are respectively equal to GH and HC. This piece of paper is similar in shape to the letter paper.

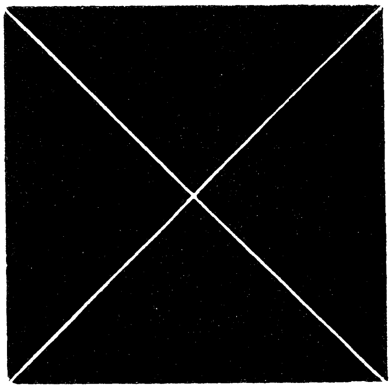
7. It can be made equal in size to the letter paper, by measuring off CE and EG equal to the sides of the latter.

8. A figure like this is called a rectangle or an oblong. By superposition, it is proved that (1) the four angles are right angles and all equal, (2) the four sides are not all equal. (3) But the two long sides are equal, and so also are the two short sides.



9. Now take this rectangular piece of paper, and fold it obliquely so that one of the short sides falls upon one of the longer sides. Then fold and remove the portion which overlaps. Unfolding the sheet, we find that it is now square, *i.e.*, its four angles are right angles, and all its sides are equal.

10. The crease which passes through a pair of the opposite



corners is a diagonal of the square. One other diagonal is obtained by folding the square through the other pair of corners.

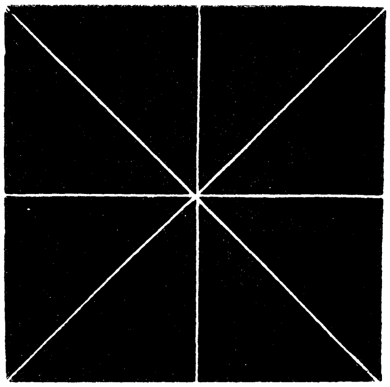
11. We see that the diagonals are at right angles to each other, and that they bisect each other.

12. The point of intersection of the diagonals is called the centre of the square.

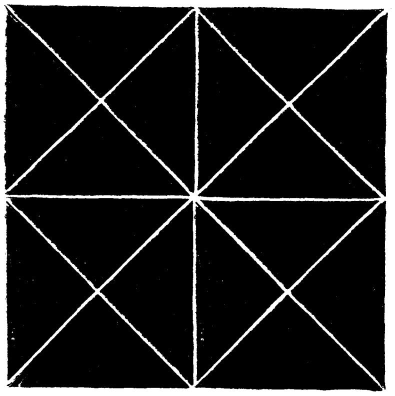
13. Each diagonal divides the square into two equal right angled isosceles triangles, whose vertices are at opposite corners.

14. The two diagonals together divide the square into four equal right-angled isosceles triangles, whose vertices are at the centre of the square.

15. Now, 'fold again, laying one side of the square upon its opposite side. We get a crease which passes through the centre of the square. It is at right anglesto the other sides and bisects them (1). It is also parallel to them (2). It is itself bisected at the centre (3). It divides the square into two equal rectangles, which are, therefore, each half of it (4) Each of these rectangles is equal to the triangles into which either diagonal divides the square (5).



16. Let us fold the square again, laying the remaining two sides one upon the other. The crease now obtained and the one referred to in para. 15 divide the square into 4 equal squares.



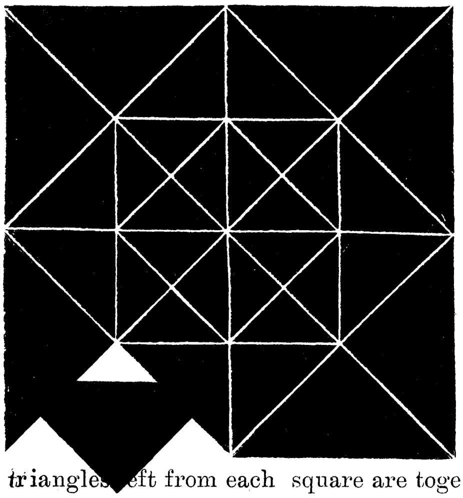
17. Folding again through the corners of the smaller squares which are at the centres of the sides of the larger square, we obtain a square which is inscribed in the latter.

18. This square is half the larger square, and has the same centre.

19. By joining the midpoints of the sides of the inner square, we obtain a square which is $\frac{1}{4}$ of the original square. By repeating the process, we can obtain any number of squares which are to one another as

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \text{\&c.,}$$

or $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\ldots$



Each square leaves $\frac{1}{2}$ of the next larger square, *i.e.*, the four triangles left from each square are together equal to half of it. The sums of all these triangles increased to any number cannot exceed the original square, and they must eventually absorb the whole of it.

Therefore $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+$ &c. to infinity $= 1$.

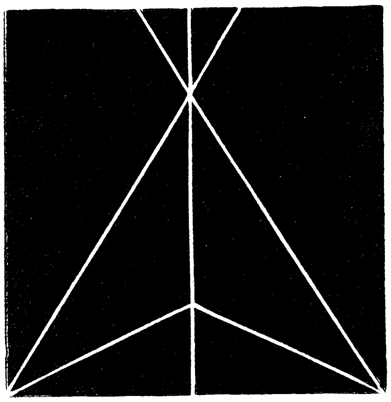
20. The centre of the square is the centre of its circumscribing and inscribed circles. The latter circle touches the sides at their mid-points, as these are nearer to the centre than any other points on the sides.

21. Any crease through the centre of the square divides it into two trapeziums which are equal in all respects. A second crease at right angles to the above divides the square into four congruent quadrilaterals, of which two opposite angles are right angles. The quadrilaterals are concyclic.

CHAPTER II.

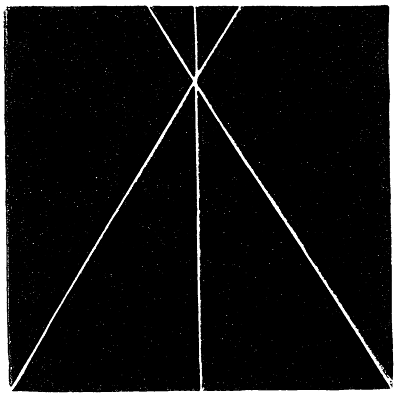
THE EQUILATERAL TRIANGLE.

Now take this square piece of paper, fold it double laying two opposite edges one upon the other. We obtain a crease which passes through the middle points of the remaining sides, and is at right angles to them. Take any point on this line, fold through it and the two corners of the square which are on each side of it. We thus get isosceles triangles, standing on a side of the square.



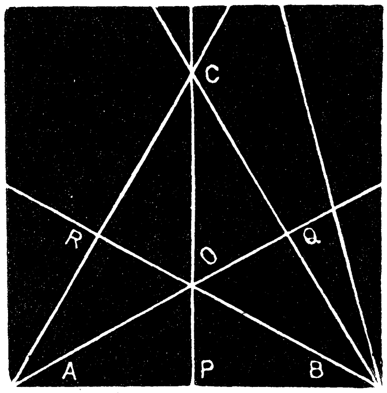
2. The middle line divides the isosceles triangle into two equal right-angled triangles.

3. The vertical angle is bisected.



4. If we so take the point on the middle line, that its distance from either corner of the square is equal to a side of it, we should obtain an equilateral triangle. This point is easily determined by turning the base through one end of it until the other end rests upon the middle line.

5. Fold the equilateral triangle by laying each of the sides upon the base. We have thus obtained the three altitudes of the triangle.



6. Each of the altitudes divides the triangle into two equal right-angled triangles.

7. They bisect the sides at right angles.

8. They pass through a common point.

9. Let the altitudes AQ and CP meet in O. Join BO and produce it to meet AC in R. Then BR can be proved to be the third altitude. From the triangles AOP and COQ, OP=OQ. From Δs OPB and OQB, $\angle$OBP=$\angle$OBQ. Again from triangles ABR and CBR, $\angle$BRA=$\angle$BRC, *i.e.*, each of them is a right angle. That is, BOR is an altitude of the equilateral triangle ABC. It also bisects AC in R.

10. It can be proved as above that OA, OB and OC are equal, and that OP, OQ and OR are also equal.

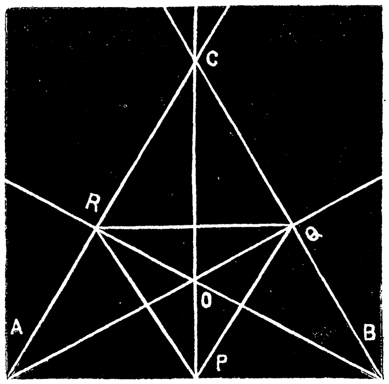
11. Circles can, therefore, be described with O as centre and passing through A, B and C and through P, Q and R. The latter circle touches the sides of the triangle.

12. The equilateral triangle ABC is divided into six equal right-angled triangles which have one set of their equal angles at O, and into three congruent symmetrical concyclic quadrilaterals.

13. The Δ AOC is double the ΔQOC; therefore, AO=2 OQ. Similarly, BO=2 OR and CO=2 OP. The radius of the circumscribing circle is twice the radius of the inscribed circle.

14. The right angle at A of the square is trisected by the straight lines AO, AR. The angle BAC $= \frac{2}{3}$ of a right angle. The $\angle$s PAO and RAO are each $\frac{1}{3}$ of a right angle. Similarly with the angles at B and C.

15. The six angles at O are each $\frac{2}{3}$ of a right angle.



16. Fold through PQ, QR, and RP. Then PQR is an equilateral triangle. It is a fourth of the triangle ABC.

17. PQ, QR & RP are each parallel to CA, AB & BC and halves of them.

18. APQR is a rhombus. So are BPRQ and CRPQ.

19. PQ, QR & RP bisect the corresponding altitudes.

20.

$$CP^2 + AP^2 = AC^2$$
$$CP^2 + \frac{1}{4}AC^2 = AC^2$$
$$CP^2 = \frac{3}{4}AC^2$$
$$CP = \frac{\sqrt{3}}{2}AC = \frac{\sqrt{3}}{2}AB = AB \times \cdot 866\ldots\ldots$$

21. The $\triangle$ ABC=rectangle under AP, CP.

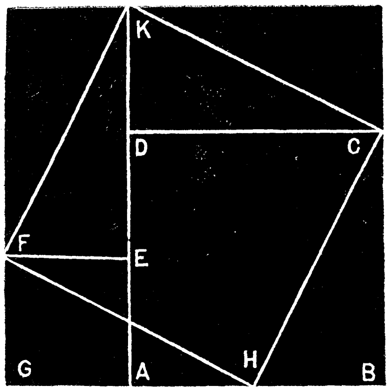
$$i.e.\ \frac{1}{2}AB. \times \frac{\sqrt{3}}{2}.AB.$$
$$= \frac{\sqrt{3}}{4}.AB^2. \quad = AB^2 \times \cdot 433\ldots\ldots$$

22. The angles of the triangle CAP are in the ratio of $1 : 2 : 3$, and its sides are in the ratio of $\sqrt{1} : \sqrt{3} : \sqrt{4}$. Pythagoras called it the most beautiful scalene triangle.

CHAPTER III.

SQUARES AND RECTANGLES.

Fold the given square as in the annexed figure. This affords the well-known proof of the 47th Proposition of the first book of Euclid. FGH being a right-angled triangle, the square on FH = the squares on FG and GH.

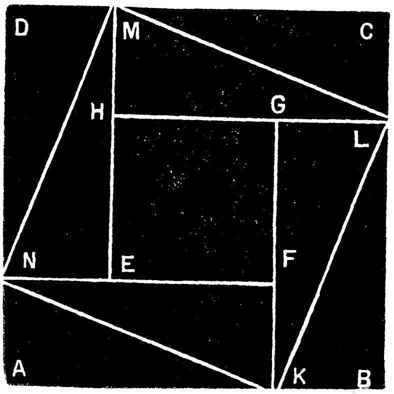


Sq. FA + sq. DB = sq. FC.

It is easily proved that FC is a square, and that the triangles FGH, HBC, KDC, and FEK are equal in every respect.

If the triangles FGH and HBC are cut and placed upon the other two triangles, the square FHCK is made up.

If AB = a, AG = b, and FH = c, $a^2 + b^2 = c^2$.



2. Fold the given square like this. Here the rectangles AF, BG, CH and DE are equal, as also the triangles of which they are composed. EFGH is a square, as also KLMN.

Let AK = a, KB = b, and NK = c.

Then $a^2 + b^2 = c^2$, *i.e.*, sq. KLMN.

The sq. ABCD = $(a+b)^2$.

Now sq. ABCD overlaps the sq. KLMN by the four triangles AKN, BLK, CML, and DNM.

But these four triangles are equal to two of the rectangles, *i.e.*, to $2ab$.

Therefore $(a+b)^2=a^2+b^2+2ab$.

3. EF$=a-b$, and sq. EFGH$=(a-b)^2$.

The square EFGH is less than the square KLMN by the 4 triangles FNK, GKL, HLM, and EMN.

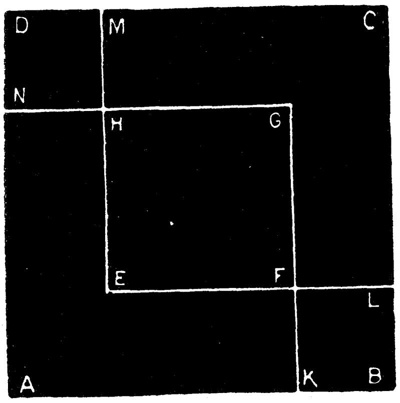
But these 4 triangles make up two of the rectangles, *i.e.*, $2ab$.

$\therefore (a-b)^2=a^2+b^2-2ab$.

4. The sq. ABCD overlaps the square EFGH by the 4 rectangles AF, BG, CH, and DE.

$\therefore (a+b)^2-(a-b)^2=4ab$.

5. In the annexed figure, the sq. ABCD$=(a+b)^2$, and the sq. EFGH $=(a-b)^2$. Also

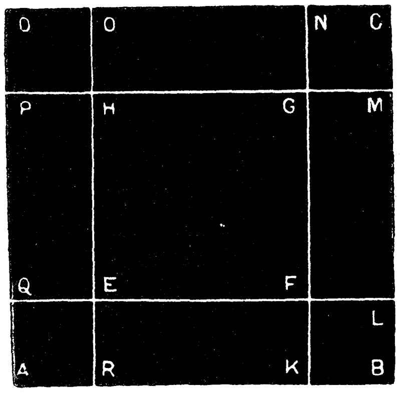


Sq. AKGN=sq. ELCM $=a^2$.

Sq. KBLF=sq. NHMD $=b^2$.

Squares ABCD and EFGH are together = the latter four squares put together

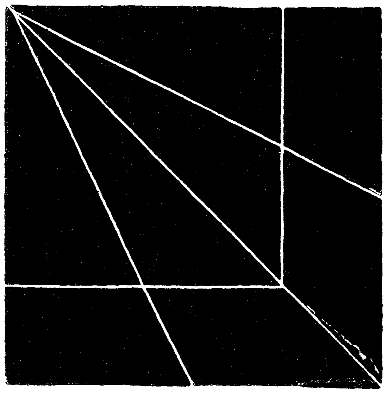
=twice the square AKGN and twice the square KBLF, that is, $(a+b)^2+(a-b)^2=2a^2+2b^2$.



6. In this figure the rectangle PL is equal to $(a+b)(a-b)$.

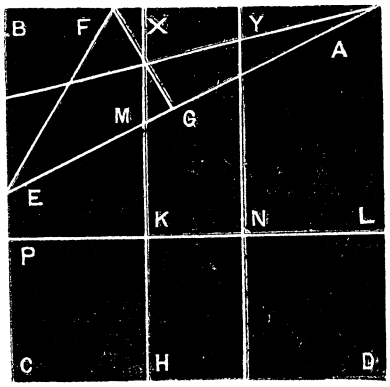
Because the rectangle EK=FM, rect. PL=sq. PK−sq. AE, *i.e.*, $(a+b)(a-b)=a^2-b^2$.

7. If squares be described about the diagonal of the given square, the right angle at one corner being common to them, the lines which join this corner with the middle points of the opposite sides of the given square bisect the corresponding sides of all the inner squares.



The angles which these lines make with the diagonal and the adjacent sides are respectively equal, and their magnitude is constant for all squares as may be seen by superposition. Therefore the midpoints of the sides of the inner squares must lie on these lines.

8. ABCD being the given square piece of paper, it is



required to obtain by folding, the point X in AB, such that the rectangle under AB, BX is equal to the square on AX.

Double BC upon itself and take its midpoint E.

Fold through E, A.

Lay EB upon EA and fold so as to get EF, FG.

Take AX=AG

Then the rectangle under AB, BX=sq. on AX.

Complete the rect. BXHC and the square AXKL.

Let XH cut EA in M. Take FY=FB.

Then FB=FG=FY=XM

and $XM = \frac{1}{2}AX$.

Now, because BY is bisected in F and produced to A

Rect. under AB.AY+sq. on FY=sq. on AF
=sq. on AG+sq. on FG.

∴ Rect. under AB, AY
=sq. on AG.
=sq. on AX.

But sq. on AX=four times sq. on XM=sq. on BY.

∴ AX=BY
and AY=BX.

∴ Rect. under AB, BX=sq. on AX.

AB is said to be divided in X in medial section.

Also
Rect. under AB, AY=sq. on BY,

i.e., AB is divided in medial section, also in Y.

9. A circle can be described with F as centre and passing through B, G and Y. It will touch EA at G, because FG is the shortest distance from F to the line EGA.

10. Rect. XYNK=sq. CHKP,

i.e., Rect. under AX, XY=sq. on AY,

i.e., AX is divided in medial section in Y.

Similarly BY is divided in medial section in X.

11. Sq. on AB+sq. on BX=three times the rectangle under AB, BX.

12. Rectangles BH, and YD being each = rect. under AB, BX, rect. HY+sq. CK=rect. under AB, BX.

13. Rect. HY = Rect. BK, *i.e.*, rect. under AX, BX = rect. under AB, XY.

14. Rect. HN=Rect. under AX, BX−sq. on BX.

15. Let AB=a, BX=x.

Then $(a-x)^2=ax$.

$a^2+x^2=3ax$.

Again,

$$x^2-3ax+a^2=0$$

$$x=\frac{a}{2}(3-\sqrt{5})$$

$$x^2=\frac{a^2}{2}\cdot(7-3\sqrt{5})$$

$$a-x=\frac{a}{2}(\sqrt{5}-1)=a\times\cdot6180.$$

$$(a-x)^2=\frac{a^2}{2}(3-\sqrt{5})=a^2\times\cdot3819\ldots\ldots$$

The rect. BXKP

$=(a-x)x$

$=a^2(\sqrt{5}-2)=a^2\times\cdot2360\ldots\ldots$

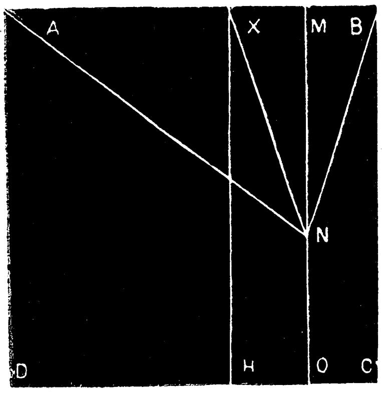
16. $EA^2=5EB^2=\frac{5}{4}AB^2$.

$$EA=\frac{\sqrt{5}}{2}AB=a\times1\cdot1180\ldots\ldots$$

17. In the language of proportion

AB : AX :: AX : BX.

The straight line AB is said to be divided "in extreme and mean ratio."



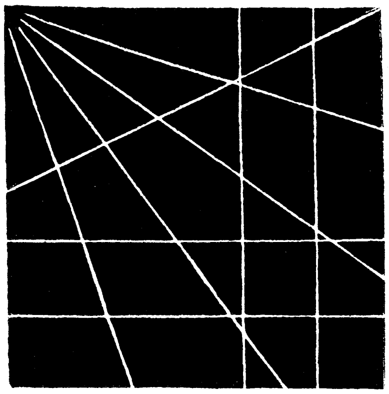
18. Let AB be divided in X in medial section. Complete the rectangle XBCH. Halve the rectangle by the line MNO. Find the point N by laying XA over XN and fold through XN, NB, and NA. Then ABN is an isosceles triangle having its angles ABN and ANB double the angle BAN.

$$AX = XN = NB$$
$$\angle ABN = \angle BXN$$
$$\angle XAN = \angle XNA$$
$$\angle BXN = 2\angle XAN$$
$$\angle ABN = 2\angle BAN.$$

$$\begin{aligned} AN^2 &= MN^2 + AM^2 \\ &= BN^2 - BM^2 + AM^2 \\ &= AX^2 + AB \,.\, AX \\ &= AB.\, BX + AB \,.\, AX \\ &= AB^2 \end{aligned}$$

$$\therefore\ AN = AB$$

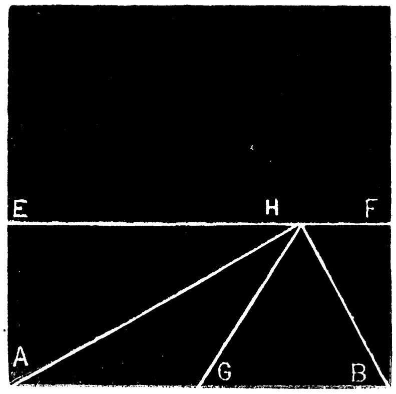
$$\angle\ BAN = \frac{2}{5} \text{ of a right angle.}$$



19. The right angle at A can be divided into five equal parts as in annexed figure.

20. To describe a right-angled triangle on the base AB, with a given altitude.

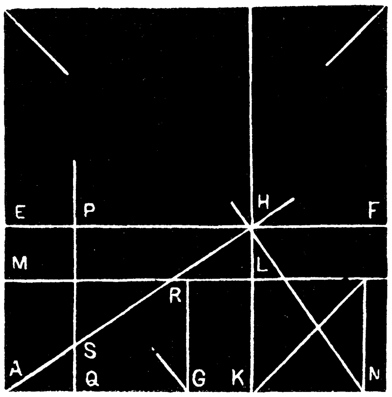
Fold EF parallel to AB at the distance of the given altitude.



Take G the middle point of AB. Find H by folding GB through G so that B may fall on EF.

Fold through H and A, G, and B.

AHB is the triangle required.



21. AKLM is a rectangle. It is required to find a square equal to it in area.

Make KN=KL.

Find G the middle point of AN.

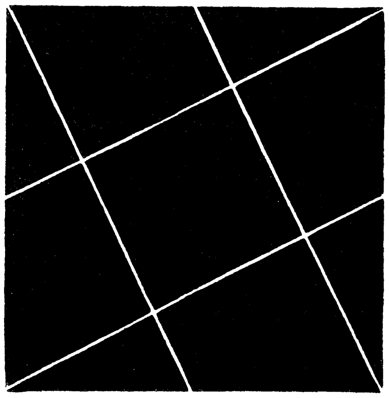
Describe the right-angled triangle AHN with the vertex on KLH.

Describe a square on KH, KHPQ.

The square is equal to the given rectangle.

22. HA and SQ divide the rectangle into 3 parts which can be fitted into the square KHPQ.

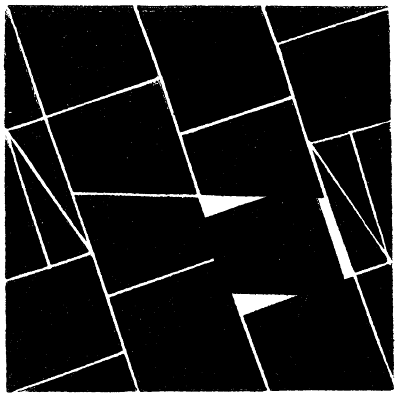
23. Take four equal squares and cut each of them into two pieces through the middle point of one of the sides and an opposite corner. Take also another equal square. The eight pieces can be arranged round the square so as to form a complete square.



This is a very interesting puzzle.

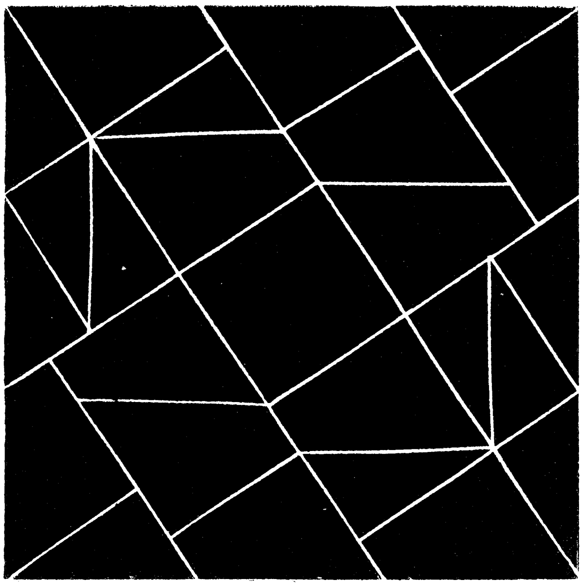
The fifth square may also be cut like the others and the puzzle put this way.

Four of the squares obviously form a complete square. Absorb the fifth square into a new square.



24. Similar puzzles can be made by cutting the squares through one corner and the trisectional points of the opposite side.

If the nearer point is taken 10 squares are required; if the remoter point is taken 13 squares are required.



25. The above puzzles are based upon the formulæ

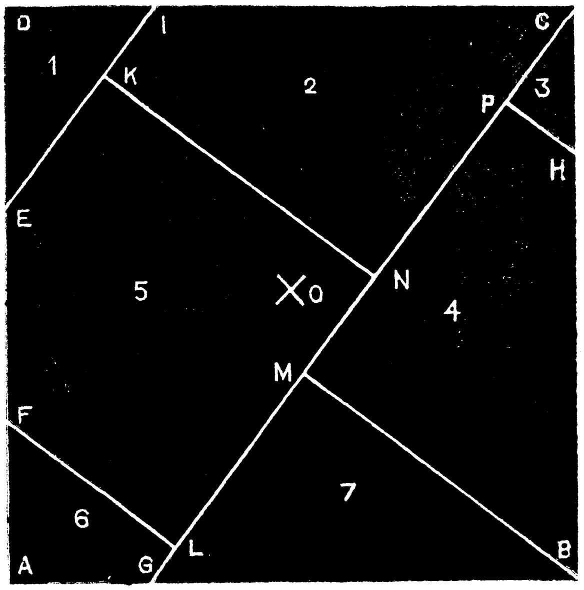
$$1^2+2^2=5$$
$$1^2+3^2=10$$
$$2^2+3^2=13.$$

26. The process may be continued, but the number of squares will become inconveniently large.

27. Consider the figure in Art. 1, Chapter III. If the four triangles at the corners of the given square are removed, one square is left. If the two rectangles FK and KG are removed, two squares in juxtaposition are left.

28. The given square may be cut into pieces which can be arranged into two squares. There are various ways of doing this. The diagram in Art. 23, Chapter III. suggests the following elegant method:—The required pieces are the square in the centre, and the four congruent symmetrical concyclic quadrilaterals at the corners. In this figure, the lines from the midpoints of the sides pass through the corners of the given square, and the central square is one-fifth of it. The magnitude of the inner square can be varied by taking other points on the sides instead of the corners.

29. The given square can be divided as follows into three equal squares:—



Take BG = half the diagonal of the square.

Fold through C and G.

Fold BM perpendicular to CG.

Take MP, CN, and NL each = BM.

Fold PH, NK, LF at right angles to CG, as in the figure.

Take NK = BM, and fold IKE at right angles to NK.

Then the pieces 1, 4 and 6, 3 and 5, and 2 and 7 form three equal squares.

Now $CG^2 = 3BG^2$

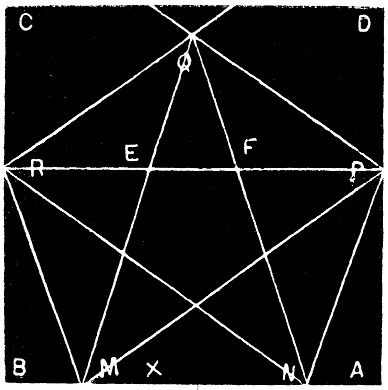
and from the triangles CBG and CMB

$$\frac{BM}{BC} = \frac{BG}{CG}$$

$$\therefore\ BM = \frac{a}{\sqrt{3}}.$$

CHAPTER IV.

THE PENTAGON.



To cut off a regular pentagon from the square ABCD.

Divide AB in X in medial section and take M the mid point of XB.

Then AB . BX = AX^2, BM=MX.

Take AN=BM or MX.

Then MN=AX.

Lay NP and MR equal to MN, so that P and R may lie on AD and BC respectively.

Lay RQ and PQ=MR and NP.

MNPQR is the pentagon required.

In fig. in para. 18, Chap. III., AN which is equal to AB, has the point N on the perpendicular MO. If A be moved on AB over the distance MB, then it is evident that N will be moved on to BC, and X to M.

Therefore in the present figure NR=AB. Similarly MP=AB. PR is also equal to AB and parallel to it.

$\angle$ BMR is $\frac{4}{5}$ of a right angle. Therefore the angle NMR= $\frac{6}{5}$ of a right angle. Similarly $\angle$ MNP is $\frac{6}{5}$ of a right angle.

From the triangles NMR and RQP, $\angle$ NMR= $\angle$ RQP=$\frac{6}{5}$ of a rt. angle.

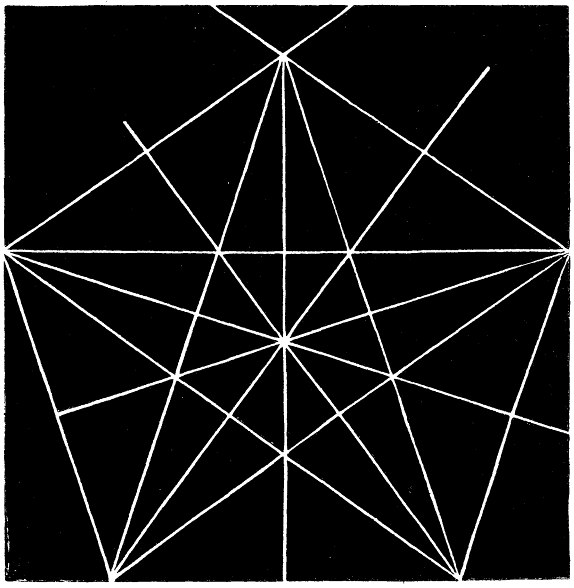
The three angles at M, N and Q of the pentagon being each equal to $\frac{6}{5}$ of a rt. ∠, the remaining 2 angles are together equal to $\frac{12}{5}$ right angles, and they are equal. Therefore each of them is $\frac{6}{5}$ of a rt. angle.

Therefore all the angles of the pentagon are equal.

It is also equilateral from the construction.

2. The base MN of the pentagon is equal to AX, *i.e.*, to $\frac{AB}{2}(\sqrt{5}-1) = AB \times \cdot 6180\ldots\ldots$

3. The greatest breadth of the pentagon is AB.



4. If p be the altitude,

$$AB^2 = p^2 + \left\{\frac{AB}{4}(\sqrt{5}-1)\right\}^2$$

$$=p^2+\mathrm{AB}^2.\frac{3-\sqrt{5}}{8}.$$

$$p^2=\mathrm{AB}^2\left\{1-\frac{3-\sqrt{5}}{8}\right\}$$

$$=\mathrm{AB}^2.\frac{5+\sqrt{5}}{8}$$

$$p=\mathrm{AB}.\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$=\mathrm{AB}\times\cdot9510\ldots\ldots=\mathrm{AB}\,\mathrm{Cos}\,18^\circ$$

5. If R be the radius of the circumscribing circle,

$$\mathrm{R}=\frac{\mathrm{AB}}{2\cos 18^\circ}.=\frac{2\mathrm{AB}}{\sqrt{10+2\sqrt{5}}}$$

$$=\mathrm{AB}\,\frac{\sqrt{5-\sqrt{5}}}{10}$$

$$=\mathrm{AB}\times\cdot5257\ldots\ldots$$

6. If r be the radius of the inscribed circle,

$$r=p-\mathrm{R}=\mathrm{AB}.\sqrt{\frac{5+\sqrt{5}}{40}}$$

$$=\mathrm{AB}\times\cdot4253\ldots\ldots$$

7. The area of the pentagon is $5r\times\frac{1}{2}$ the base of the pentagon,

$$i.e.,\ 5\mathrm{AB}.\sqrt{\frac{5+\sqrt{5}}{40}}.\frac{\mathrm{AB}}{4}(\sqrt{5}-1)$$

$$=\mathrm{AB}^2.\frac{5}{4}.\sqrt{\frac{5-\sqrt{5}}{10}}=\mathrm{AB}^2\times\cdot6571\ldots\ldots$$

8. In fig. in para. 1, Chap. IV., let PR be divided by MQ and NQ in E and F.

$$\text{Then } RE=FP=\frac{MN}{2}\cdot\frac{1}{\cos 36^\circ}=AB\cdot\frac{\sqrt{5}-1}{\sqrt{5}+1}$$

$$=AB.\ \frac{3-\sqrt{5}}{2}\cdot\ \ldots\ldots\ldots\ldots(1)$$

$$EF=AB-2\,RE=AB-AB(3-\sqrt{5})=AB\,(\sqrt{5}-2)\ldots\ldots(2)$$

$$RF=MN.$$

$$RF:RE::RE:EF\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\sqrt{5}-1:3-\sqrt{5}::3-\sqrt{5}:2\sqrt{5}-4\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The area of the inner pentagon

$$=EF^2.\ \frac{5}{4}\sqrt{\frac{5-\sqrt{5}}{10}}$$

$$=AB^2.\ (\sqrt{5}-2)^2.\ \frac{5}{4}\cdot\ \sqrt{\frac{5-\sqrt{5}}{10}}$$

$$=AB^2.\ (9-4\sqrt{5}).\ \frac{5}{4}\cdot\ \sqrt{\frac{5-\sqrt{5}}{10}}\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The larger pentagon : the smaller $::1:(\sqrt{5}-2)^2$

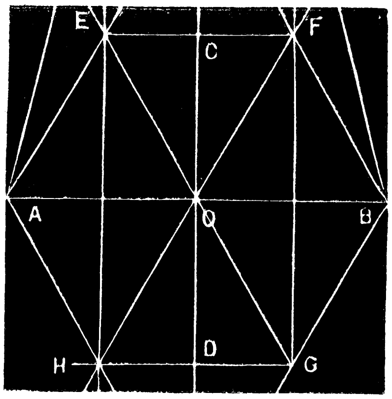
$$::1:\ \cdot05569\ldots\ldots$$

9. If in the figure in Art. 1, Chapter IV, angles QEK and QFL are made equal to EQR or FQP, K, L being points on the sides QR and QP respectively, then EFLQK will be a regular pentagon equal to the inner pentagon. Pentagons can be similarly described on the remaining sides of the inner pentagon. The resulting figure consisting of six pentagons is very elegant.

CHAPTER V.

THE HEXAGON.

To cut off a regular hexagon from the given square.



Fold through the mid points of the opposite sides, and obtain the lines AOB and COD.

On both sides of AO and OB describe equilateral triangles, AEO, AHO; BFO and BGO.

Join EF and HG.

AEFBGH is a regular hexagon.

It is unnecessary to give the proof.

2. The greatest breadth of the hexagon is AB.

3. The altitude of the hexagon is

$$\frac{\sqrt{3}}{2}\,\text{AB.} = \text{AB} \times \cdot 866\ldots\ldots$$

4. If R be the radius of the circumscribing circle,

$$\text{R} = \frac{1}{2}\,\text{AB.}$$

5. If r be the radius of the inscribed circle,

$$r = \frac{\sqrt{3}}{4}\,\text{AB} = \text{AB} \times \cdot 433\ldots\ldots$$

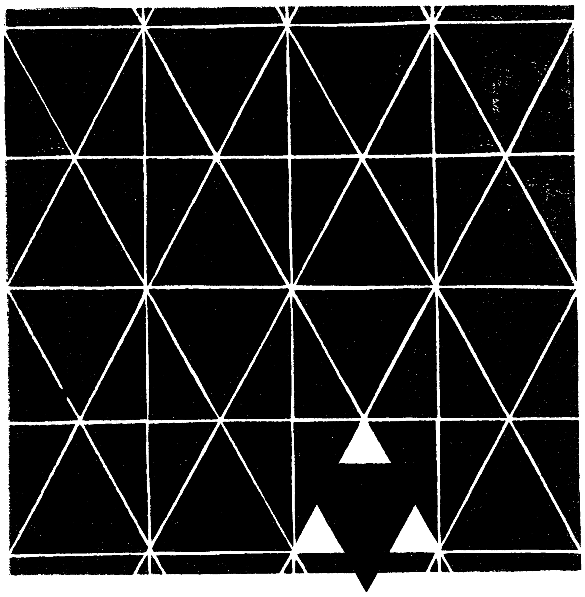
6. The area of the hexagon is 6 times the area of the triangle HOG.

$$= 6.\,\frac{\text{AB}}{4}.\,\frac{\sqrt{3}}{4}\,\text{AB.}$$

$$= \frac{3\sqrt{3}}{8}\text{AB}^2. = \text{AB}^2 \times \cdot 6495\ldots\ldots$$

Also the hexagon $= \frac{3}{4}$. AB. CD.

$= 1\frac{1}{2}$ times the equilateral triangle on AB.

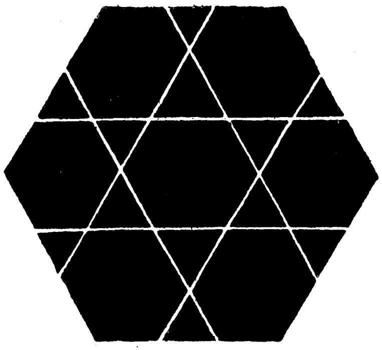


7. The above figure is an instance of ornamental folding, into equilateral triangles and hexagons.

8. A hexagon is formed from an equilateral triangle by folding the three corners to the centre.

The side of the hexagon is $\frac{1}{3}$ of the side of the equilateral triangle.

The area of the hexagon $= \frac{2}{3}$ of the equilateral triangle.

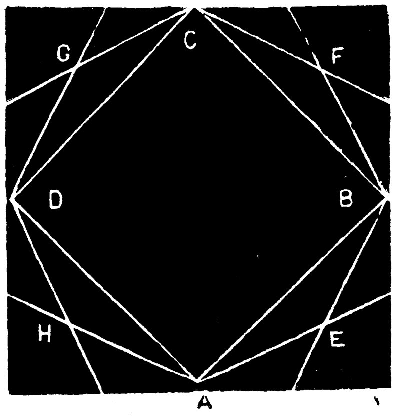


9. The hexagon can be divided into equal regular hexagons and equilateral triangles as in the annexed figure by folding through the points of trisection of the sides.

CHAPTER VI.

THE OCTAGON.

To cut off a regular *octagon* from the given square.



Obtain the inscribed square by joining the mid-points A, B, C, D of the sides of the given square.

Bisect the angles which the sides of the inscribed square make with the sides of the other. Let the bisecting lines meet in E, F, G and H.

EFGH is a *regular octagon*.

The Δs ABE, BCF, CDG and DAH are equal isosceles triangles. The octagon is therefore *equilateral*.

The angles at the vertices E, F, G, H of the same four Δs are each one right angle and a half: because the angles at the base are each one-fourth of a right angle.

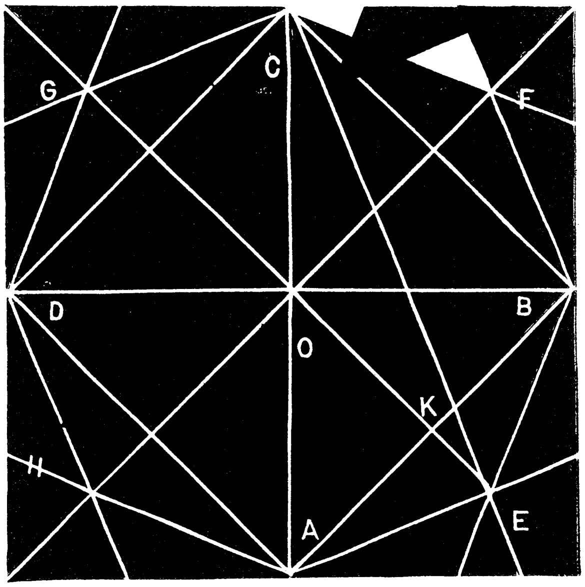
Therefore the angles of the octagon at A, B, C and D are each one right angle and a half.

Thus the octagon is also *equiangular*.

2. The greatest breadth of the octagon is the side of the given square, a.

3. If R be the radius of the circumscribed circle,

$$R = \frac{a}{2}.$$



4. The angle subtended at the centre by each of the sides is *half* a rt. angle.

5. Join OE and let it cut AB in K.

Then $AK = OK = \frac{OA}{\sqrt{2}} = \frac{a}{2\sqrt{2}}$

$KE = OA - OK = \frac{a}{2} - \frac{a}{2\sqrt{2}} = \frac{a}{4}(2 - \sqrt{2})$

Now from the Δ AKE, $AE^2 = AK^2 + KE^2$

$$= \frac{a^2}{8} + \frac{a^2}{8} \cdot (3 - 2\sqrt{2})$$

$$= \frac{a^2}{8}(4 - 2\sqrt{2})$$

$$= \frac{a^2}{4} \cdot (2-\sqrt{2})$$

$$\therefore \text{AE} = \frac{a}{2}\sqrt{2-\sqrt{2}}.$$

6. The altitude of the octagon is CE.

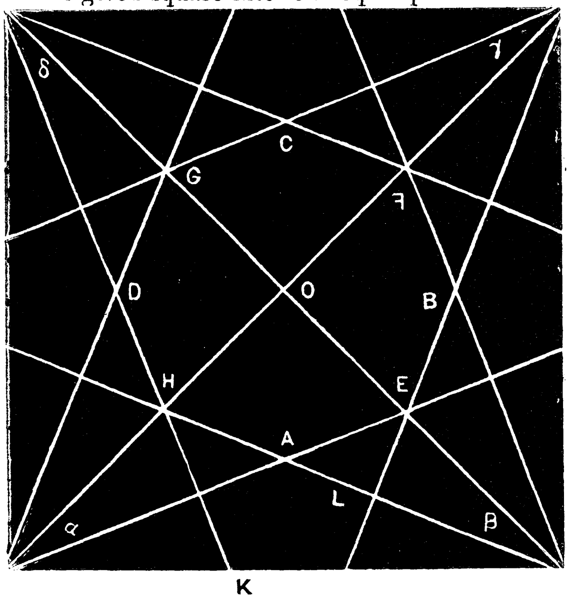
But $\text{CE}^2 = \text{AC}^2 - \text{AE}^2$

$$= a^2 - \frac{a^2}{4}(2-\sqrt{2}) = \frac{a^2}{4}(2+\sqrt{2})$$

$$\therefore \text{CE} = \frac{a}{2}\sqrt{2+\sqrt{2}}\cdot$$

7. The area of the octagon is eight times the triangle AOE and equals 4 OE. AK $= 4. \frac{a}{2} \cdot \frac{a}{2\sqrt{2}} \cdot = \frac{a^2}{\sqrt{2}} \cdot$

8. A regular octagon may also be obtained by dividing the angles of the given square into four equal parts.



It is easily seen that $E\delta = \alpha\delta = a$;

$$\beta\delta = a\sqrt{2};$$

$$\beta E = a(\sqrt{2}-1);$$

$$\beta E = \alpha H = \alpha K;$$

$$K\beta = a - a(\sqrt{2}-1)$$

$$= a(2-\sqrt{2}).$$

Now $K\delta^2 = a^2 + a^2(\sqrt{2}-1)^2 = a^2(4-2\sqrt{2})$

$\therefore$ $K\delta = a\sqrt{4-2\sqrt{2}}$.

Also $GE = \beta\delta - 2\beta E$

$$= a\sqrt{2} - 2a(\sqrt{2}-1)$$

$$= a(2-\sqrt{2});$$

$\therefore$ $HO = \frac{a}{2}(2-\sqrt{2})$.

Again $O\delta = \frac{a}{2}\sqrt{2}$;

and $H\delta^2 = HO^2 + O\delta^2$

$$= \frac{a^2}{4}\{6 - 4\sqrt{2} + 2\}$$

$$= a^2(2-\sqrt{2});$$

$\therefore$ $H\delta = a\sqrt{(2-\sqrt{2})}$.

$HK = K\delta - H\delta$

$$= a\sqrt{4-2\sqrt{2}} - a\sqrt{2-\sqrt{2}}$$

$$= a.\,(\sqrt{2-\sqrt{2}}).\,(\sqrt{2}-1)$$

$$= a\,\sqrt{10-7\sqrt{2}}.$$

$$\therefore \text{ AL} = \frac{1}{2}\text{HK} = \frac{a}{2}\sqrt{10-7\sqrt{2}};$$

and $\text{HA} = \frac{a}{2}\sqrt{20-14\sqrt{2}}.$

9. The area of the octagon is 8 times the triangle HOA.

$$=4.\ \text{HO}.\ \frac{\text{HO}}{\sqrt{2}}$$

$$=\text{HO}^2.\ 2\sqrt{2}$$

$$=\left\{\frac{a}{2}(2-\sqrt{2})\right\}^2.\ 2\sqrt{2}$$

$$=\frac{a^2}{4}.\ 2\sqrt{2}.\ (6-4\sqrt{2})$$

$$=a^2.\ (3\sqrt{2}-4)$$

$$=a^2.\ \sqrt{2}.\ (\sqrt{2}-1)^2.$$

10. This octagon : the octagon in para. 1

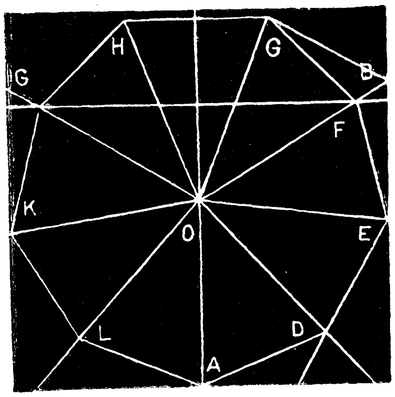
$$::\ (2-\sqrt{2})^2 : 1 \text{ or } 2 : (\sqrt{2}+1)^2;$$

and their bases are to one another as $\sqrt{2} : \sqrt{2}+1$.

CHAPTER VII.

THE NONAGON.

Any angle can be trisected fairly accurately by paper folding.



Obtain the three equal angles at the centre of an equilateral triangle.

For convenience of folding, cut out the three angles, AOB, BOC and COA.

Trisect each of the angles as in the figure, and make the arms$=$OA.

The trisection can be facilitated by first describing a circle with O as centre and radius OA.

2. The angles of a nonagon are each $\frac{14}{9}$ of a rt. angle$=140°$.

The angle subtended by each side at the centre is $\frac{4}{9}$ of a rt. angle or $40°$.

Half this angle is $\frac{1}{7}$ of the angle of the nonagon.

3. $OA=\frac{1}{2}a$.

This is also the radius of the circumscribing circle, R.

The radius of the inscribed circle$=$R. cos $20°=\frac{1}{2}a\cos 20°$.

$$=\frac{a}{2}\times \cdot 9396926$$

$$=a\times \cdot 4698463.$$

The area of the nonagon is 9 times the triangle AOL

$$= 9 \,.\, \mathrm{R} \,.\, \tfrac{1}{2}\, \mathrm{R}\ \mathrm{Sin}\ 40^\circ$$

$$= \frac{9}{2}\, \mathrm{R}^2.\ \mathrm{Sin}\ 40^\circ$$

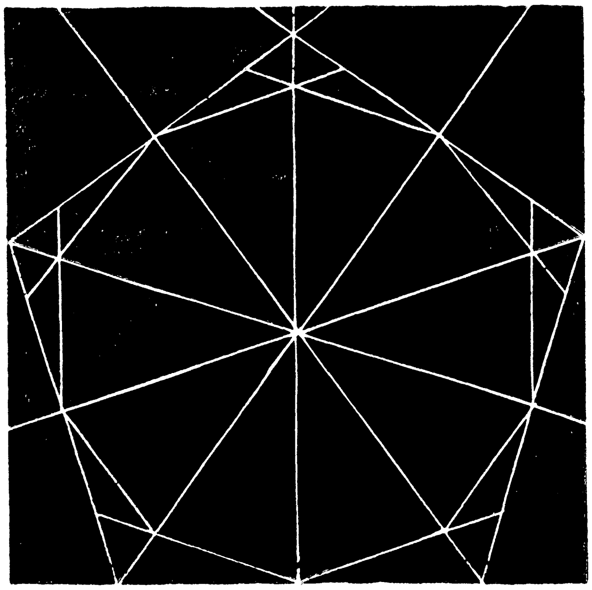
$$= \frac{9a^2}{8} \times {\cdot}6427876$$

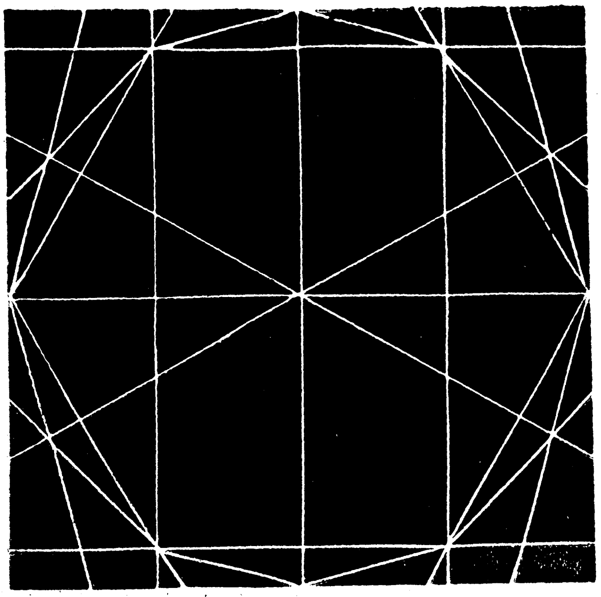
$$= a^2 \times {\cdot}723136.$$

CHAPTER VIII.

THE DECAGON AND THE DODECAGON.

THE following figures show how a regular *decagon*, and a regular *dodecagon*, can be obtained from a pentagon and hexagon respectively.

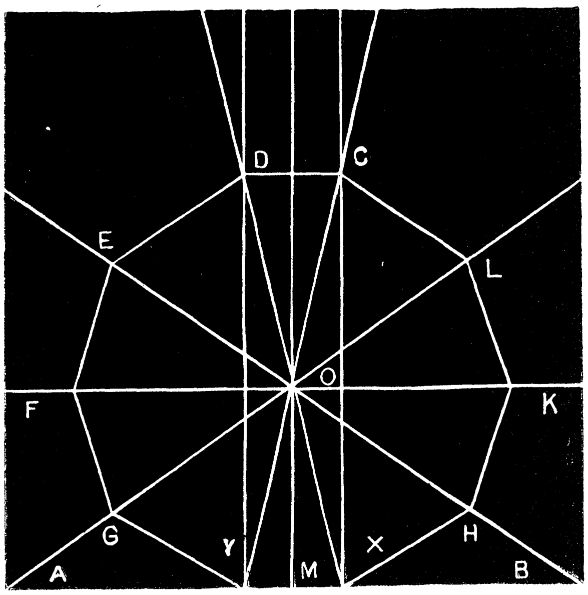




2. The main part of the process is to obtain the angles at the centre.

3. In fig. 1, the radius of the inscribed circle of the pentagon is taken for the radius of the circumscribing circle of the decagon, in order to keep it within the square.

4. A regular decagon may also be obtained as follows :



Obtain X, Y, as in Chap. III., para. 8, dividing AB in medial sections.

Take M the midpoint of AB.

Fold XC, MO, YD at right angles to AB.

Take O in MO such that YO=AY, or XO=XB.

Let YO, and XO produced meet XC, and YD in C and D respectively.

Divide the angles XOC and YOD into 4 parts by HOE, KOF, and LOG.

Take OH, OK, OL, OE, OF and OG equal to OY or OX.

Join X, H, K, L, C, D, E, F, G and Y, in order.

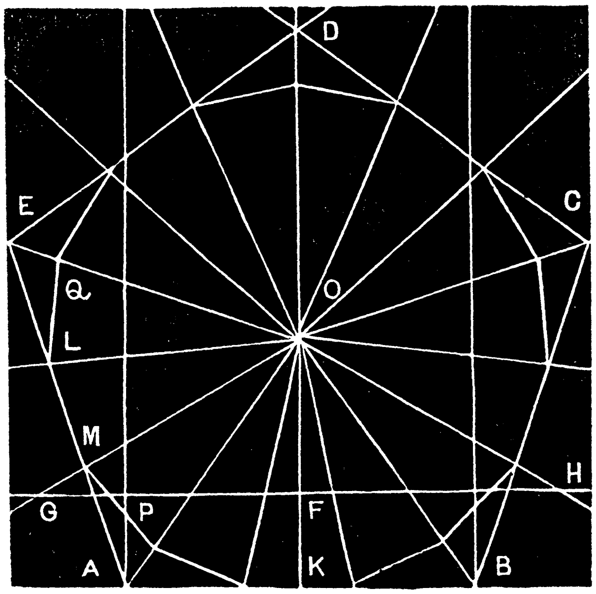
As in Chap. III., para. 17,

$$\angle XOY=\frac{2}{5} \text{ of a right angle}=36°.$$

CHAPTER IX.

THE QUINDECAGON.

THIS figure shows how the quindecagon is obtained from the pentagon.



Let ABCDE be the pentagon and O its centre.

Join OA, OB, OC, OD and OE. Produce DO to meet AB in K.

Take OF $= \frac{1}{2}$ of OD.

Fold GFH at right angles to OF. Make OG, OH $=$ OD.

Then GDH is an equilateral triangle, and the angles DOG and DOH are each 120°.

But ∠ DOA is 144°; therefore ∠ GOA is 24°.

That is, the angle AOE which is 72° is *trisected* by OG.

Bisect the ∠ GOE by OL meeting EA in L, and let OG cut EA in M;

then OL=OM.

In OA and OE take OP and OQ equal to OL or OM.

Then PM, ML, and LQ are three sides of the *quindecagon.*

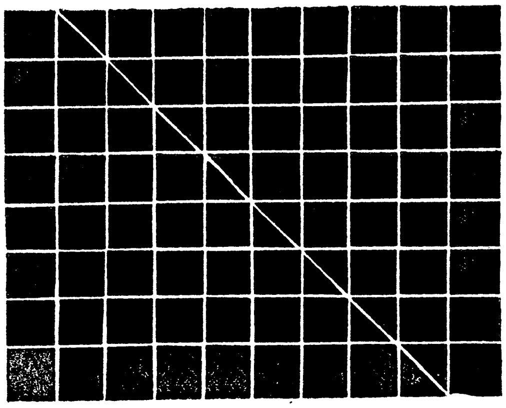
Treating similarly the angles AOB, BOC, COD, and DOE, we obtain the remaining sides of the quindecagon.

CHAPTER X.

THE PROGRESSIONS.

ARITHMETICAL PROGRESSION.

THE annexed diagram exemplifies *Arithmetical Progression.*



The horizontal lines to the left of the diagonal, including the upper and lower edges are in A.P. The initial line being a and b the common difference, the series is a, $a+b$, $a+2b$, $a+3b$, &c.

2. The portions of the horizontal lines to the right of the diagonal are also in A.P., but are in reverse order and decrease with a common difference.

3. If, generally, l be the last term, and S the sum of the series, the above diagram graphically proves the formula

$$S=\frac{n}{2}(l+a).$$

4. If a and c are two alternate terms, the middle term is

$$\frac{1}{2}(a+c).$$

5. To insert n means between a and l, the vertical line has to be folded into $n + 1$ equal parts. The common difference will be $\frac{l-a}{n+1}$.

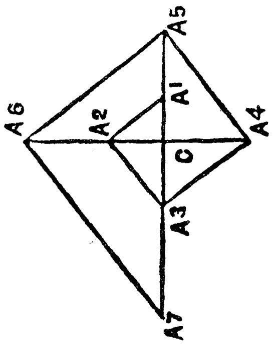
6. Considering the reverse series and interchanging a and l, the series becomes

$$a,\ a-b,\ a-2b,\ \ .\ \ .\ \ l.$$

The terms will be positive so long as $a > (n-1)b$, and thereafter they will be negative.

Geometrical Progression.

7. In a right-angled triangle, the perpendicular from the vertex on the hypotenuse is a geometric mean between the segments of the hypotenuse. Hence, if two alternate or consecutive terms of a G.P. be given in length, the series can be determined as in the accompanying figure. Here CA_1, CA_2, CA_3, CA_4, and CA_5, are in G.P., the common ratio being $\frac{CA_2}{CA_1}$.



8. If CA_1 be the unit of length, the series consists of the natural powers of the common ratio.

9. Representing the series by a, ar, ar^2,...............

$A_1 A_2 = a\sqrt{1+r^2}$.
$A_2 A_3 = ar\sqrt{1+r^2}$.
$A_3 A_4 = ar^2\sqrt{1+r^2}$.
. . . . .

These lines also form a G.P., with the common ratio r.

10. The terms can also be found backwards, in which case the common ratio will be a proper fraction. If CA_5 be the unit, CA_4 is the common ratio. The sum of the series to infinity is $\frac{CA_5}{CA_5 - CA_4}$.

11. In the manner described above, one Geometrical mean can be found between two given lines, and by continuing the process, 3, 7, 15, &c., means can be found. In general, 2^n-1 means can be found, n being any positive integer.

12. It is not possible to find two Geometrical means between two given lines, merely by folding through known points. In the above figure, CA_1 and CA_4 being given, it is required to find A_2 and A_3. Take two rectangular pieces of paper, and so arrange them, that their outer edges lie on A_1 and A_4, and a corner of each lies on the straight lines CA_2 and CA_3, while at the same time the other edges ending in the said corners coincide. The positions of the corners determine CA_2 and CA_3.

13. This process gives the cube root of a given number, for if CA_1 be the unit, the series is $1, r, r^2, r^3$.

14. There is a very interesting legend in connection with this problem. "The Athenians when suffering from the great plague of eruptive typhoid fever in 430 B.C., consulted the oracle at Delos as to how they could stop it. Apollo replied that they must *double* the *size* of his altar which was in the form of a *cube*. Nothing seemed more easy, and a new altar was constructed having each of its *edges* double that of the old one. The God, not unnaturally indignant, made the pestilence worse than before. A fresh deputation was accordingly sent to Delos, whom he informed that it was useless to trifle with him, as he must have his altar exactly doubled. Suspecting a mystery, they applied to the Geometricians. Plato, the most illustrious of them, declined the task, but referred them to Euclid, who had made a special study of the problem." Euclid's name is an interpellation for that of Hippocrates. Hippocrates reduced the question to that of finding two means between two straight lines, one of which is twice as long as the other. If a, x, y and $2a$ be the terms of the series $x^3 = 2a^3$. He did not, however, succeed in finding the means. Menæchmus, a pupil of Plato, who lived between 375 and 325 B.C., gave the following two solutions :

$$a : x :: x : y :: y : 2a.$$

From this relation we obtain the following three equations:

$$x^2 = ay \quad \ldots\ldots\ldots\ldots \quad (1)$$
$$y^2 = 2ax \quad \ldots\ldots\ldots\ldots \quad (2)$$
$$xy = 2a^2 \quad \ldots\ldots\ldots\ldots \quad (3)$$

(1) and (2) are equations of *parabolas* and (3) is the equation of a *rectangular hyperbola*. Equations (1) and (2) as well as (1) and (3) give $x^3 = 2a^3$. The problem was solved by taking the intersection (α) of the two parabolas (1) and (2) and (β) of the parabola (1) with the rectangular hyperbola (3).

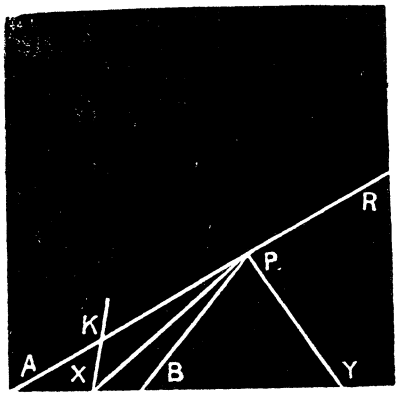
HARMONIC PROGRESSION.

15. Fold any lines AR, PB as in the next figure, P being on AR, and B on the edge of the paper. Fold again so that AP and PR may both coincide with PB. Let PX, PY be the creases thus obtained, X and Y being on AB.

Then the points A, X, B, Y form an harmonic range. That is, AB is divided internally in X and externally in Y such that

$$\text{AX} : \text{BX} :: \text{AY} : \text{BY}.$$

16. It is evident that every line cutting PA, PX, PB and PY will be harmonically divided.



17. Having given A, B and X to find Y : fold any line XP and mark K corresponding to B. Fold AKPR, and BP. Bisect the angle BPR by PY by folding through P so that PB and PR may coincide.

Because XP bisects the angle APB,

$$\therefore AX : BX :: AP : BP,$$
$$:: AY : BY.$$

18. $AX : BX :: AY : BY$

or $AY—XY : XY—BY :: AY : BY.$

Thus, AY, XY, and BY are in Harmonic Progression, and XY is the Harmonic Mean between AY and BY.

Similarly AB is the H.M. between AX and AY.

19. If YB and YX be given, to find the third term YA, we have only to describe any right angled triangle on XY as hypotenuse and make $\angle XPA = \angle XPB$.

20. Let $AX=a$, $AB=b$, and $AY=c$.

$$\text{Then } b=\frac{2ac}{a+c};$$

$$\text{or, } ab+bc=2ac$$

$$\text{or, } c=\frac{ab}{2a-b}=\frac{b}{2-\frac{b}{a}}.$$

When $a=b$, $c=b$.

When $b=2a$, $c=\infty$.

Therefore when X is the middle point of AB, Y is at an infinite distance to the right of B. It approaches B as X approaches it, and ultimately the 3 points will coincide.

As X moves from the middle of AB to the left, Y moves from an infinite distance on the left towards A, and ultimately X, A, and Y coincide.

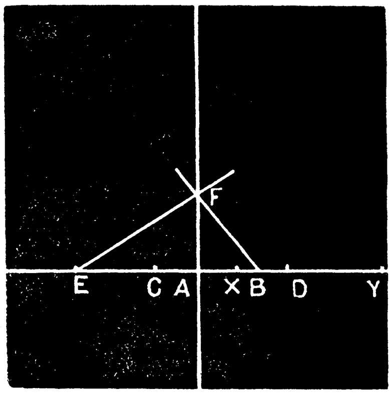
21. If E be the middle point of AB,

$$EX.EY = EA^2 = EB^2$$

for all positions of X and Y with reference to A or B.

Each of the two systems of pairs of points X and Y is called a system in *Involution*, the point E being called the *centre* and A or B the *focus* of the system. The two systems together may be regarded as one system.

22. AX and AY being given, B can be found as follows:—



Produce XA and take AC=AX.

Take D the middle point of CY.

Take CE=AD or AE=CD.

Fold through A so that AF may be at right angles to CAY.

Find F such that DF=DC.

Fold through EF and obtain FB, such that FB is at right angles to EF.

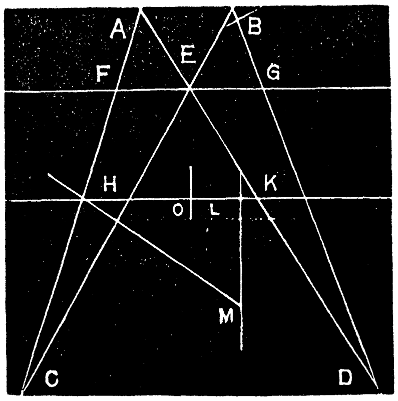
CD is the Arithmetical Mean between AX and AY.

AF is the Geometrical Mean between AX and AY.

AF is also the Geometrical Mean between CD or AE and AB.

Therefore AB is the Harmonic Mean between AX and AY.

23. The following is a very simple method of finding the H. M. between two given lines.



Take AB, CD on the edges of the square equal to the given lines. Fold the diagonals AD, BC and the sides AC, BD. Fold through E the point of intersection of the diagonals so that FEG may be at right angles to the other sides of the square or parallel to AB and CD. Let FEG cut AC and BD in F and G. Then FG is the H. M. between AB and CD.

For

$$\frac{EF}{AB} = \frac{CE}{CB}$$

$$\text{and } \frac{EG}{CD} = \frac{BE}{CB}$$

$$\therefore \frac{EF}{AB} + \frac{EG}{CD} = \frac{CE}{CB} + \frac{BE}{CB} = 1.$$

$$\therefore \frac{1}{AB} + \frac{1}{CD} = \frac{1}{EF} = \frac{2}{FG}.$$

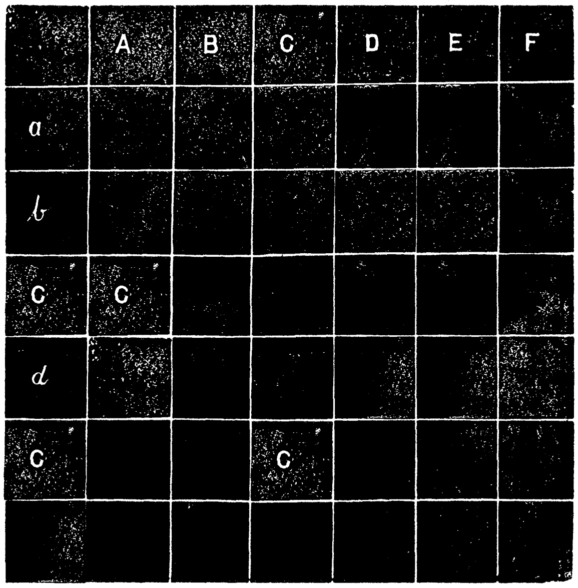
24. The join HK of the midpoints of AC and BD is the A. M. between AB and CD.

25. To find the G. M. take HL in HK=FG. Fold LM at right angles to HK. Take O the midpoint of HK and find M in LM so that OM=OH. HM is the G M. between AB and CD as well as between FG and HK. The G. M. between two quantities is the G. M. between their A. M. and H. M.

SUMMATION OF CERTAIN SERIES.

26. To sum up the series

$$1+3+5\ldots\ldots+(2n-1)$$



Divide the given square into a number of equal squares as in the accompanying figure. Here we have 49 squares, but the number may be increased as we please.

The number of squares will evidently be a square number, the square of the number of divisions of the sides of the given square.

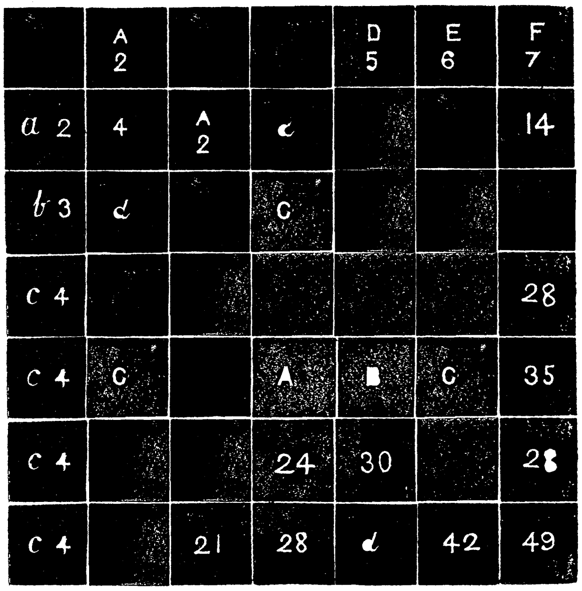
Let each of the small squares be considered as the unit.

The numbers of unit squares in each of the gnomons A*a*, B*b*, &c., are respectively 3, 5, 7, 9, 11, 13.

Therefore the sum of the series 1, 3, 5, 7, 9, 11, 13 is 7^2.

Generally, $1+3+5+\ldots+(2n-1)=n^2$.

27. To find the sum of the cubes of the first n natural numbers.



Fold the square into 49 equal squares as in the preceding article, and letter the gnomons. Fill up the squares with numbers as in the multiplication table.

The number in the initial squares is $1=1^3$.

The sums of the numbers in the gnomons Aa, Bb, &c., are 2^3, 3^3, 4^3, 5^3, 6^3 and 7^3.

The sum of the numbers in the first horizontal row is the sum of the first seven natural numbers. Let us call it S.

Then the sums of the numbers in rows *a*, *b*, *c*, *d*, &c., are

2S, 3S, 4S, 5S, 6S, and 7S.

Therefore the sum of all the numbers is

$$S\,(1+2+3+4+5+6+7)=S^2.$$

Therefore, the sum of the cubes of the first seven natural numbers is equal to the square of the sum of those numbers.

Generally, $1^3+2^3+3^3\ldots\ldots+n^3$

$$=(1+2+3\ldots\ldots+n)^2$$

$$=\left\{\frac{n(n+1)}{2}\right\}^2$$

or $(n.\overline{n+1})^2-(\overline{n-1}.n)^2\equiv(n^2+n)^2-(n^2-n)^2\equiv 4n^3$.

Putting $n=1, 2, 3$ ....in order, we have

$$4.1^3=(1.2)^2-(0)^2$$
$$4.2^3=(2.3)^2-(1.2)^2$$
$$4.3^3=(3.4)^2-(2.3)^2$$
$$\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$
$$4.n^3=(n.\overline{n+1})^2-(\overline{n-1}\;n)^2.$$

Adding up

$$4S=\{\,n(n+1)\,\}^2$$

$$\therefore\;\; s=\left\{\frac{n(n+1)}{2}\right\}^2.$$

28. If S_n be the sum of the first n natural numbers,

$$S_n{}^2-S_n{}^2{}_{-1}=n^3.$$

29. To sum the series

$$1.2+2.3+3.4\ldots\ldots+(n-1).n.$$

In the above table, the figures on the diagonal commencing from 1, are the squares of the natural numbers in order.

The figures in one gnomon can be subtracted from the corresponding figures in the succeeding gnomon. By this process we obtain

$$n^3-(n-1)^3=n^2-(n-1)^2$$
$$+2\{\,n(n-1)+(n-2)+(n-3)\ldots\ldots+1\,\}$$
$$=n^2+(n-1)^2+2\{\,1+2\ldots\ldots+\overline{n-1}\,\}$$

$$=n^2+(n-1)^2+n(n-1)$$

$$=(n-\overline{n-1})^2+3(n-1)n$$

$$=1+3(n-1)n.$$

Now $n^3-(n-1)^3=1+3(n-1)n$

$$(n-1)^3-(n-2)^3=1+3(n-2)(n-1)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2^3-1^3=1+3.2.1$$

$$1^3-0^3=1+0.$$

Hence, by addition,

$$n^3=n+3\{1.2+2.3+\ldots\ldots+(n-1).n\}$$

$$\therefore\ 1.2+2.3\ldots\ldots+(n-1).n=\frac{n^3-n}{3}=\frac{(n-1)n(n+1)}{3}.$$

30. To find the sum of the squares of the first n **natural** numbers,

$$1.2+2.3\ldots\ldots+(n-1).n$$

$$=2^2-2+3^2-3\ldots\ldots+n^2-n$$

$$=1^2+2^2+3^2\ldots\ldots+n^2-(1+2+3\ldots+n)$$

$$=1^2+2^2+3^2\ldots\ldots+n^2-\frac{n(n+1)}{2}.$$

$$1^2+2^2+3^2\ldots\ldots+n^2=\frac{(n-1)n(n+1)}{3}+\frac{n(n+1)}{2}$$

$$=n(n+1)\left\{\frac{n-1}{3}+\frac{1}{2}\right\}$$

$$=\frac{n(n+1)(2n+1)^*}{6}$$

31. To sum up the series

$$1^2+3^2+5^2\ldots\ldots+(2n-1)^2$$

$$n^3-(n-1)^3=n^2+(n-1)^2+n(n-1)$$

$$=(2n-1)^2-(n-1)n.$$

$* \; 6n^2\equiv n(n+1)(2n+1)-(n-1)n(2n-1)$

Put $n=1, 2, 3, 4\ldots\ldots$ in order and add up.

Thus by putting $n=1, 2, 3$............

$$1^3-0^3=1^2-0$$

$$2^3-1^3=3^2-1.2$$

$$3^3-2^3=3^2-2.3$$

....

....

$$n^3-(n-1)^3=(2n-1)^2-(n-1).n.$$

Adding up, we get

$$n^3=1^2+3^2+5^2......+(2n-1)^2$$
$$-\{1.2+2.3+3.4 .....+(n-1).n\}$$

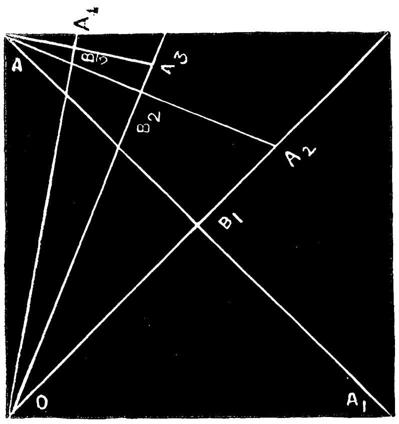
$$\therefore 1^2+3^2+5^2......+(2n-1)^2$$
$$=n^3+\frac{n^3-n}{3}$$
$$=\frac{4n^3-n}{3}=\frac{n(2n-1)(2n+1)}{3}.$$

CHAPTER XI.

POLYGONS.

Take O the centre of the square and its diameters. Bisect the right angles at the centre, then the half right angles, and so on. Then we obtain 2^n equal angles round the centre and the magnitude of each of the angles is $\frac{4}{2^n}$ of a right angle, n being a positive integer. Mark off equal lengths on each of the lines which radiate from the centre. If the extremities of the radii are joined successively, we get regular polygons of 2^n sides.

2. Let us find the perimeter and area of these polygons. In the accompanying figure let OA and OA_1 be two radii at right angles to each other. Let the radii OA_2, OA_3, OA_4, &c., divide the right angle AOA_1 in 2, 4, 8 ...... parts. Join AA_1, AA_2, AA_3 ......cutting the radii OA_2, OA_3, OA_4 ... at B_1, B_2, B_3 ... respectively, at right angles. Then B_1, B_2, B_3 ... are the mid points of the respective chords. Then AA_1, AA_2, AA_3, AA_4, ... are the sides of the inscribed polygons of 2^2, 2^3, 2^4 ... sides respectively, and OB_1, OB_2 .... are the respective apothegms.



Let OA=R,

$a(2^n)$ represent the *side* of the inscribed polygon of 2^n sides, $b(2^n)$ the corresponding *apothegm*, $p(2^n)$ its *perimeter*, and $A(2^n)$ its *area*.

For the *square*,

$$a(2^2)=R\sqrt{2}\,;\ p(2^2)=R.2^2.\sqrt{2}\,;$$

$$b(2^2)=\frac{R}{2}\sqrt{2};\ A(2^2)=R^2.2.$$

For the *octagon*,

in the two triangles AB_2O and AB_1A_2

$$\frac{AB_2}{B_1A_2}=\frac{OA}{AA_2}$$

$$\therefore\ \ \tfrac{1}{2}\ A\ A_2{}^2=R.B_1A_2=R\{R-b(2^2)\}$$

$$=R\left\{R-\frac{R}{2}\sqrt{2}\right\}=\tfrac{1}{2}R^2.\ (2-\sqrt{2})$$

$$\text{or}\ \ AA_2=R\sqrt{2-\sqrt{2}}=a(2^3)\ \ \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$p(2^3)=R.\ 2^3\sqrt{2-\sqrt{2}}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$b(2^3)=OB_2=\sqrt{OA^2-AB_2{}^2}=\sqrt{R^2\left(1-\frac{2-\sqrt{2}}{4}\right)}$$

$$=\sqrt{\frac{R^2(2+\sqrt{2})}{4}}=\tfrac{1}{2}R\sqrt{2+\sqrt{2}}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$A(2^3)=\tfrac{1}{2}\ \text{perimeter}\ \times\ \text{apothegm}$$

$$=R.2^2.\sqrt{2-\sqrt{2}}.\tfrac{1}{2}R\sqrt{2+\sqrt{2}}=R^2.2\sqrt{2}.$$

Similarly for the polygon of 16 sides,

$$a(2^4)=R\sqrt{2-\sqrt{2+\sqrt{2}}}\,;$$

$$p(2^4)=R.2^4.\sqrt{2-\sqrt{2+\sqrt{2}}}\,;$$

$$b(2^4)=\frac{R}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}\,;$$

$$A(2^4)=R^2.2^2.\ \sqrt{2-\sqrt{2}}\,;$$

and for the polygon of 32 sides,

$$a(2^5)=R\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}};$$

$$t(2^5)=\frac{R}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}};$$

$$p(2^5)=R.2^5.\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$A(2^5)=R^2.2^3\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

The general law is thus clear.

Also, $$A(2^n)=\frac{R}{2}\cdot p(2^{n-1}).$$

When the number of sides is increased indefinitely the apothegm becomes obviously equal to the radius. Thus the limit of

$$\sqrt{2+\sqrt{2+\sqrt{2}}}\ldots\ldots \text{is } 2.^*$$

3. If perpendiculars are drawn to the radii at their extremities, we get regular polygons circumscribing the circle and also the polygons described as in the preceding article, and of the same number of sides.

In the next figure, let AE be a side of the inscribed polygon and FG a side of the circumscribed polygon.

Then from the triangles FIE and EIO,

$$\frac{OE}{OI}=\frac{FE}{EI}=\frac{FG}{AE};$$

$$\therefore\ FG=R.\frac{AE}{OI}.$$

The values of AE, and OI being known by the previous article, FG is found by substitution.

The areas of the two polygons are to one another as $FG^2:AE^2$, *i e.*, as $R^2:OI^2$.

---

* If x represent the limit, $x=\sqrt{2+x}$, a quadratic which gives $x=2$, or -1; the latter value is, of course, inadmissible.

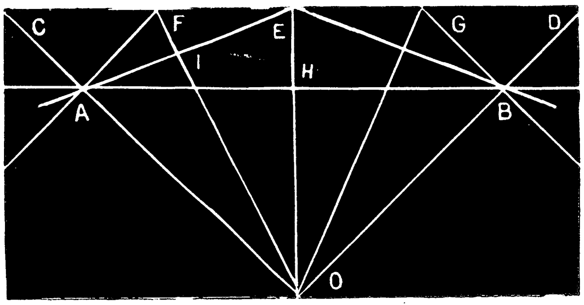
4. In the preceding articles it has been shown how regular polygons can be obtained of $2^2, 2^3 \ldots 2^n$ sides. And if a polygon of m sides be given, it is easy to obtain polygons of $2^n.m$ sides.

5. In the annexed figure, AB and CD are respectively the sides of the inscribed and circumscribed polygons of n sides. Take E the midpoint of CD and join AE, BE. AE and BE are the sides of the inscribed polygon of $2n$ sides.

Fold AF, BG at right angles to AC and BD, meeting CD in F and G.

Then FG is a side of the circumscribed polygon of $2n$ sides.

Join OF, OG and OE.



Let p, P be the perimeters of the inscribed and circumscribed polygons respectively of n sides, and A, B their areas, and p', P′ the perimeters of the inscribed and circumscribed polygons respectively of $2n$ sides, and A′, B′ their areas.

Then

$$p = n.\mathrm{AB},\ \mathrm{P} = n.\mathrm{CD},\ p' = 2n.\mathrm{AE},\ \mathrm{P}' = 2n.\mathrm{FG}.$$

Because OF bisects the ∠ COE and AB is parallel to CD,

$$\frac{\mathrm{CF}}{\mathrm{FE}} = \frac{\mathrm{CO}}{\mathrm{OE}} = \frac{\mathrm{CO}}{\mathrm{AO}} = \frac{\mathrm{CD}}{\mathrm{AB}};$$

$$\therefore \quad \frac{\text{CE}}{\text{FE}} = \frac{\text{CD}+\text{AB}}{\text{AB}};$$

$$\text{or} \quad \frac{4n.\text{CE}}{4n.\text{FE}} = \frac{n.\text{CD}+n.\text{AB}}{n.\text{AB}};$$

$$\therefore \quad \frac{2\text{P}}{\text{P}'} = \frac{\text{P}+p}{p};$$

$$\therefore \quad \text{P}' = \frac{2\text{P}p}{\text{P}+p}.$$

Again, from the similar triangles EIF and AHE,

$$\frac{\text{EI}}{\text{AH}} = \frac{\text{EF}}{\text{AE}},$$

$$\text{or} \quad \text{AE}^2 = 2\text{AH.EF};$$

$$\therefore \quad 4n^2.\text{AE}^2 = 4n^2.\text{AB.EF},$$

$$\text{or} \quad p' = \sqrt{\text{P}'p}.$$

Now, $\text{A} = 2n\triangle\text{AOH}, \ \text{B} = 2n\triangle\text{COE}$

$\text{A}' = 2n\triangle\text{AOE}, \ \text{B}' = 4n\triangle\text{FOE}.$

The triangles AOH and AOE are of the same altitude,

$$\therefore \quad \frac{\triangle\text{AOH}}{\triangle\text{AOE}} = \frac{\text{OH}}{\text{OE}}.$$

Similarly,

$$\frac{\triangle\text{AOE}}{\triangle\text{COE}} = \frac{\text{OA}}{\text{OC}}.$$

Again because AB is parallel to CD, $\frac{\text{OH}}{\text{OE}} = \frac{\text{OA}}{\text{OC}}$

$$\therefore \quad \frac{\triangle\text{AOH}}{\triangle\text{AOE}} = \frac{\text{AOE}}{\text{COE}}$$

$$\therefore \quad \frac{\text{A}}{\text{A}'} = \frac{\text{A}'}{\text{B}} \quad \text{or} \quad \text{A}' = \sqrt{\text{AB}}.$$

To find B′, because the triangles COE and FOE have the same altitude, and OF bisects the angle COE,

$$\frac{\triangle\text{COE}}{\triangle\text{FOE}} = \frac{\text{CE}}{\text{FE}} = \frac{\text{OC}+\text{OE}}{\text{OE}},$$

$$\text{and } OE = OA,$$

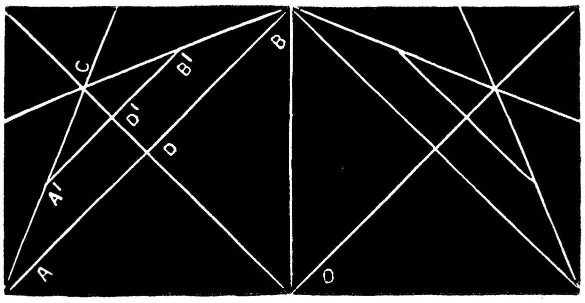
$$\text{and } \frac{OC}{OA} = \frac{OE}{OH} = \frac{\triangle AOE}{\triangle AOH};$$

$$\therefore \frac{\triangle COE}{\triangle FOE} = \frac{\triangle AOE + \triangle AOH}{\triangle AOH}.$$

Multiplying both sides by $4n$, we get $\frac{2B}{B'} = \frac{A' + A}{A}$;

$$\therefore B' = \frac{2AB}{A + A'}.$$

6. Given the radius R and apothegm r of a regular polygon, to find the radius R′ and apothegm r' of a regular polygon of the *same perimeter* but of *double* the number of sides.



Let AB be a side of the first polygon, O its centre, OA the radius of the circumscribed circle, and OD the apothegm. On OD produced take OC=OA or OB. Join AC, BC. Fold OA′ and OB′ perpendicular to AC and BC respectively. Join A′ B′ cutting OC in D′. Then the chord A′ B′ is half of AB, and the angle A′OB′ is half of AOB. OA′ and OD′ are respectively the radius R′ and apothegm r' of the second polygon.

Now OD′ is the arithmetical mean between OC and OD and OA′ is the mean proportional between OC and OD′.

$$\therefore r' = \tfrac{1}{2}(R + r), \text{ and } R' = \sqrt{Rr'}.$$

7. Now, take on OC, OE = OA and join A′E

Then A′D′ being less than A′C, and ∠ D′A′C being bisected by A′E,

$$\text{ED}' \text{ is less than } \frac{1}{2}\,\text{CD}', \textit{ i.e., } \text{less than } \frac{1}{4}\,\text{CD}$$

$$\therefore\ \text{R}_1 - r_1 \text{ is less than } \frac{1}{4}\,(\text{R} - r).$$

As the number of sides is increased, the polygon approaches the circle of the same perimeter, and R and r become equal to the radius of the circle.

That is,

$$\text{R} + r + \text{R}_1 - r_1 + \text{R}_2 - r_2 + \ldots\ldots\ldots\ldots\ldots$$

$$= \text{the diameter of the circle} = \frac{p}{\pi}.$$

Also,

$$\text{R}_1{}^2 = \text{R}r_1 \text{ or } \text{R}.\,\frac{r_1}{\text{R}_1} = \text{R}_1$$

$$\text{and } \frac{r_2}{\text{R}_2} = \frac{\text{R}_2}{\text{R}_1}, \text{ and so on.}$$

Multiplying both sides

$$\text{R}.\,\frac{r_1}{\text{R}_1}.\,\frac{r_2}{\text{R}_2}.\,\frac{r_3}{\text{R}_3}\ldots\ldots\ldots = \text{the radius of the circle} = \frac{p}{2\pi}.$$

8. The radius of the circle lies between R_n and r_n, the sides of the polygon being 4.2^n in number; and π lies between $\frac{2}{r_n}$ and $\frac{2}{\text{R}_n}$. The numerical value of π can therefore be calculated to any required degree of accuracy by taking a sufficiently large number of sides.

The following are the value of the radii and apothegms of the regular polygons of 4, 8, 16,...2048 sides.

4. $r = 0{\cdot}500000$ $\text{R} = r\sqrt{2} = 0{\cdot}707107$
8. $r_1 = 0{\cdot}603553$ $\text{R}_1 = 0{\cdot}653281$

……………………………………………

2048. $r_9 = 0{\cdot}636620$ $\text{R}_9 = 0\ 636620$

$$\pi = \frac{2}{0{\cdot}636620} = 3{\cdot}14159\ldots\ldots\ldots$$

9. If R″ be the radius of a regular isoperimetrical polygon of $4n$ sides

$$R''^2 = \frac{R'^2\ (R + R')}{2R}$$

or in general

$$\frac{R_{k+1}}{R_k} = \sqrt{\frac{1 + \dfrac{R_k}{R_{k-1}}}{2}}.$$

10. The radii R_1, R_2,……successively diminish, and the ratio $\frac{R_2}{R_1}$ is less than unity and equal to the *cosine* of a certain angle a

$$\frac{R_3}{R_2} = \sqrt{\frac{1 + \cos a}{2}} = \cos\frac{a}{2}$$

$$\therefore\ \frac{R_{k+1}}{R_k} = \cos\frac{a}{2^{k-1}}$$

multiplying together the different ratios, we get

$$R_{k+1} = R_1 \,.\, \cos a \,.\, \cos\frac{a}{2},\ \cos\frac{a}{2^2}\ \ldots\ldots\cos\frac{a}{2^{k-1}}.$$

The limit of $\cos a \cos\frac{a}{2^2}\ \ldots\cos\frac{a}{2^{k-1}}$, when $k = \infty$ is $\frac{\sin 2a}{2a}$ a result known as *Euler's Formula.*

11. It was demonstrated by Karl Friedrich Gauss (1777—1855) that the only regular polygons which can be constructed by elementary geometry are those the number of whose sides is $2^m(2^n + 1)$ where m and n are positive integers and $2^n + 1$ is a prime number. The first two numbers of this description are 5 and 17. We shall show here how polygons of 5 and 17 sides can be described.

The following theorems are required :—

(1) If C and D are two points on a semi-circumference

ACDB, and if C′ be the image of C with respect to AB, and R the radius of the circle,

$$AC.BD=R.(C'D-CD)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{i.}$$

$$AD.BC=R.(C'D+CD)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{ii.}$$

$$AC.BC=R.CC'\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{iii.}$$

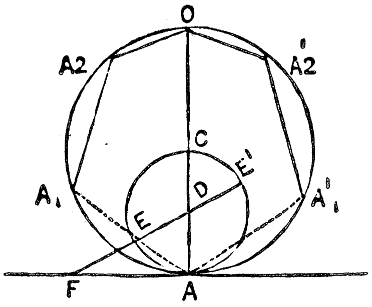
(2) Let the circumference of a circle be divided into an odd number of equal parts, and let AO be the diameter through one of the points of section A and the midpoint O of the opposite arc. Let the points of section on each side of the diameter be named A_1, A_2, A_3 .... A_n and A'_1, A'_2, A'_3 .... A'_n beginning next to A.

$$\text{Then } OA_1.\ OA_2.\ OA_3.\ldots\ OA_n=R^n.\ \text{iv.}$$

$$\text{and } OA_1.\ OA_2.\ OA_4.\ldots\ OA_n=R^{\frac{n}{2}}.$$

12. It is evident that if the chord OA_n is determined, the angle AOA_n is found and it has only to be divided into 2^n equal parts, to obtain the other chords.

13. Let us first take the *pentagon.*



By theorem iv.

$$OA_1.\ OA_2=R^2.$$

By theorem i.

$$R(OA_1-OA_2)=OA_1.\ OA_2=R^2$$

$$\therefore\ OA_1-OA_2=R$$

$$\therefore\ OA_1=\frac{R}{2}(\sqrt{5}+1)$$

and $OA_2=\frac{R}{2}(\sqrt{5}-1)$.

Hence the following construction.

Take the diameter ACO, and draw the tangent AF. Take D the midpoint of the radius OC and $AF=OC$.

On OC as diameter describe the circle CE′AE.

Join FD cutting the inner circle in E and E′.

Then $FE'=OA_1$, and $FE=OA_2$.

14. Let us now consider the polygon of seventeen sides.

*Here $OA_1 . OA_2 . OA_3 . OA_4 . OA_5 . OA_6 . OA_7 . OA_8=R^8$.

$OA_1 . OA_2 \; OA_4 . OA_8=R^4$.

and $OA_3 . OA_5 . OA_6 . OA_7=R^4$.

By theorems i. and ii.

$$OA_1 . OA_4=R\,(OA_3+OA_5)$$
$$OA_2 . OA_8=R\,(OA_6-OA_7)$$
$$OA_3 . OA_5=R\,(OA_2+OA_8)$$
$$OA_6 . OA_7=R\,(OA_1-OA_4)$$

Suppose

$$OA_3+OA_5=M,\quad OA_6-OA_7=N,$$
$$OA_2+OA_8=P,\quad OA_1-OA_4=Q.$$

Then $MN=R^2$ and $PQ=R^2$.

Again by substituting the values of M, N, P and Q in the formulæ

$$MN=R^2,\ PQ=R^2$$

and applying theorems i and ii. we get

$$(M-N)-(P-Q)=R$$

Also by substituting the values of M, N, P and Q in the above formula and applying theorems i. and ii. we get

$$(M-N)\,(P-Q)=4R^2.$$

Hence $M-N$, $P-Q$, M, N, P and Q are determined.

Again $OA_2+OA_8=P$

$OA_2 . OA_8=RN$.

Hence OA_8 is determined.

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* The principal steps are given. For a full exposition see Catalan's Theoremes et Problemes de Geometrie Elementaire.

15. By solving the equations we get

$$M-N=\tfrac{1}{2}R\,(1+\sqrt{17})$$

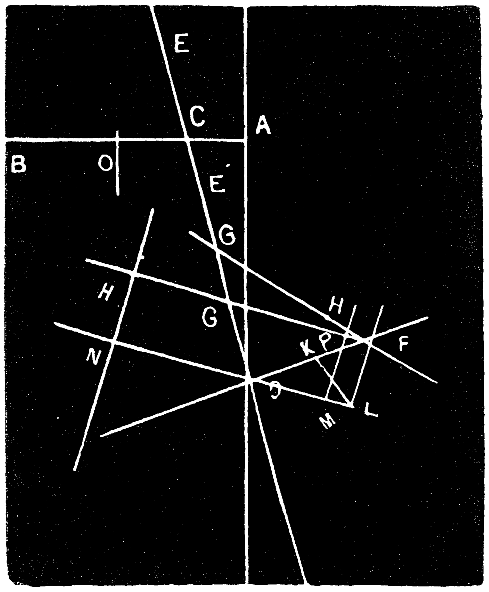
$$P-Q=\tfrac{1}{2}R\,(-1+\sqrt{17})$$

$$P=\tfrac{1}{4}R\,(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}})$$

$$N=\tfrac{1}{4}R\,(-1-\sqrt{17}+\sqrt{34+2\sqrt{17}})$$

$$OA_8=\tfrac{1}{8}R\,\Big[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}$$
$$-2\sqrt{17+3\sqrt{17}+\sqrt{170-26\sqrt{17}-4\sqrt{34+2\sqrt{17}}}}\Big]$$
$$=\tfrac{1}{8}R\,\Big[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}$$
$$-2\sqrt{17+3\sqrt{17}-\sqrt{170+38\surd}}$$

16. The geometrical construction is as follows :



Let BA be the diameter of the given circle; O its centre Bisect OA in C. Draw AD at right angles to OA and take $AD = AB$. Join CD. Take E and E′ in CD and on each side of it so that $CE = CE' = CA$.

Bisect ED in G and E′D in G′. Draw DF perpendicular to CD and take $DF = OA$.

Join FG and FG′

Take H in FG and H′ in FG′ produced so that $GH = EG$ and $G'H' = G'D$.

Then it is evident that

$DE = M - N$
$DE' = P - Q$
also $FH = N \because (DE + FH)\, FH = DF^2 = R^2$
$FH' = P \because (FH' - DE')\, FH = DF^2 = R^2$

Again in DF take K such that $FK = FH$

Draw KL perpendicular to DF and take L in KL such that FL is perpendicular to DL.

Then $FL^2 = DF.\ FK = RN$.

Again

Draw H′N perpendicular to FH′
and take $H'N = FL$.
Draw NM perpendicular to NH′.
Find M in NM such that H′M is perpendicular to FM.
Draw MP′ perpendicular to FH′.
Then $P'H'.\ FP' = P'M^2 = FL^2$
$= RN$
But $FP' + P'H' = P$
$\therefore\ P'F = OA_8$

CHAPTER XII.

GENERAL PRINCIPLES.

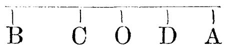
In the preceding pages we have adopted several processes, *e.g.*, bisecting and trisecting finite lines, bisecting rectilineal angles and dividing them into other equal parts, drawing perpendiculars to a given line, &c. Let us now examine the theory of these processes.

2. The general principle is one of *congruence.* Figures and straight lines are said to be congruent, if they are *identically* equal, or equal *in all respects.*

In doubling a piece of paper upon itself, we obtain the straight edges of two planes coinciding with each other. This line may also be regarded as the intersection of two planes if we consider their position during the process of folding.

In dividing a finite straight line or angle into a number of equal parts, we obtain a number of congruent parts. Equal lines and equal angles are congruent.

3. Let AB be a given finite line, divided into any two parts in C. Take O the midpoint by doubling the line on itself. Then OC is half the difference between AC and BC. Double AB and take D in AO corresponding to C. Then CD is the difference between AC and BC and it is bisected in O. As C is taken nearer to O, CO diminishes and at the same time CD diminishes at twice the rate. This property is made use of in finding the midpoint of a line by means of the compasses.



4. The above observations apply also to an angle. The line of bisection is found easily by the compasses by taking the point of intersection of two circles.

5. In the line BOA, segments to the right of O may be considered *positive* and the segments to the left of O may be considered *negative.* That is, a point moving from O to A moves positively, and a point moving in the opposite direction OB moves negatively.

$$\begin{aligned} DA &= OA - OD. \\ OC &= -OB - (-CB \\ &= -OB + CB \\ &= -(OB - CB). \end{aligned}$$

6. If OA, one arm of an angle AOP be fixed and OP be considered to revolve round O, the angles which it makes with OA are of different magnitudes. All such angles formed by OP revolving in the direction opposite to that of the hands of a watch are regarded *positive.* The angles formed by OP revolving in an opposite direction are regarded *negative.*

7. After one revolution, OP coincides with OA. Then the angle described may be called an *angle of rotation* *= four right angles. When OP has completed half the revolution, it is in a line with OAB. Then the angle described may be called an *angle of continuation* *= two right angles. When OP has completed quarter of a revolution, it is perpendicular to OA. All right angles are equal in magnitude. So are the angles of continuation and revolution.

8. Two lines at right angles to each other form four congruent quadrants. Two lines otherwise inclined form four angles, of which two vertically opposite ones are congruent.

9. The position of a point in a plane is determined by its distance from each of two lines taken as above. The distance from one line is measured parallel to the other. In Analytical Geometry, the properties of plane figures are investigated by this method. The two lines are called *axes;* the distances of the point from the *axes* are called *co-ordinates*, and the intersection of the axes is called the *origin.* This method was invented by Descartes in 1637 A.D. It has greatly helped modern research.

10. If AOB and COD be two axes, distances measured in the direction of OA, *i.e.*, to the right of COD are positive, while distances measured to the left of COD are negative. Similarly with reference to AOB, distances measured in the direction of OC are positive, while distances measured in the direction of OD are negative.

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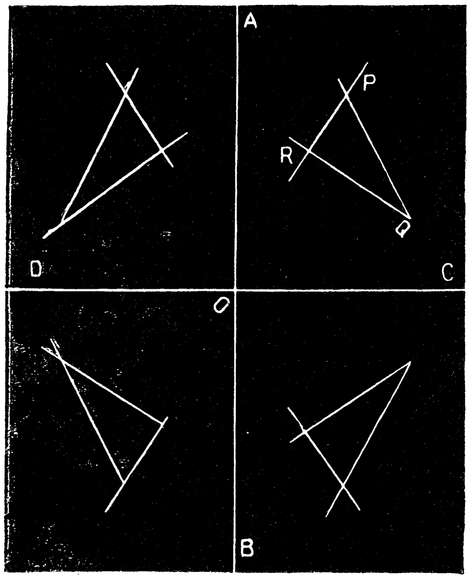
* These terms are adopted by Olaus Henrici, Ph.D., F.R.S.

11. *Axial symmetry* is defined thus:—If two figures in the same plane can be made to coincide by turning the one about a fixed line in the plane through an angle of continuation, the two figures are said to be symmetrical with regard to that line as *axis of symmetry*.

12. *Central symmetry* is thus defined:—If two figures in the same plane can be made to coincide by turning the one about a fixed point in that plane through an angle of continuation, the two figures are said to be symmetrical with regard to that point as *centre of symmetry*.

In the first case the revolution is outside the given plane, while in the second it is in the same plane.

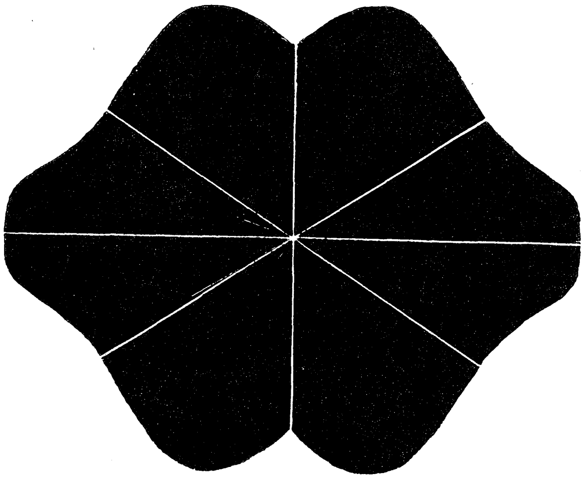
If in the above two cases, the two figures are halves of one figure, the whole figure is said to be symmetrical with regard to the axis or centre—these are called *axis* or *centre of symmetry* or simply *axis* or *centre*.



13. Now, in the quadrant AOC make a triangle PQR. Obtain its image in the quadrant COB by folding on the axis COD. Again obtain images of the two triangles in the fourth and third quadrants. It is seen that the angles in adjacent quadrants possess *axial* symmetry, while the triangles in alternate quadrants possess *central* symmetry.

14. Regular polygons of an odd number of sides possess *axial* symmetry, and regular polygons of an even number of sides possess *central* symmetry as well.

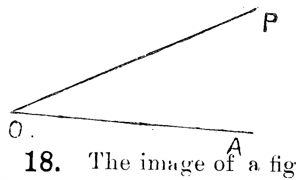
15. If a figure has two axes of symmetry at right angles to each other, the point of intersection of the axes is a centre of symmetry. This obtains in regular polygons of an even number of sides and certain curves, such as the circle, ellipse, hyperbola, and the lemniscate; regular polygons of an odd number of sides may have more axes than one, but no two of them will be at right angles to each other. If a sheet of paper is folded double and cut, we obtain a piece which has *axial* symmetry, and if it is cut fourfold, we obtain a piece which has *central* symmetry as well.



16. Parallelograms have a centre of symmetry. A quadrilateral of the form of a kite, or a trapezium with two opposite sides equal and equally inclined to either of the remaining sides, have also a centre of symmetry.

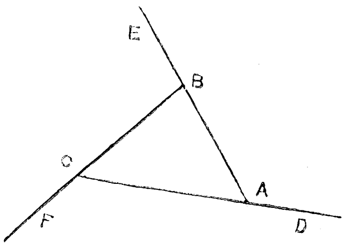
17. The position of a point in a plane is also determined by its distance from a fixed point and the inclination of the line joining the two points to a fixed line drawn through the fixed point.

If OA be the fixed line and P the given point, the length PO and ∠ AOP, determine the position of P. O is the *pole*, OA is the *prime-vector*, OP the *radius vector* and ∠ AOP the *vectorial angle*. OP and ∠ AOP are called *polar co-ordinates* of P.



18. The image of a figure symmetrical to the axis OA may be obtained by folding through the axis OA. The radii vectors of corresponding points are equally inclined to the axis.

19. Let ABC be a triangle. Produce the sides CA, AB, BC to D, E, F respectively. Suppose a person to stand at A with face towards D and then to proceed from A to B, B to C, and C to A. Then he successively describes the angles DAB, EBC, FCD. Having come to his original position A, he has completed an angle of rotation, *i.e.*, four right angles. The three exterior angles are thus together equal to four right angles.



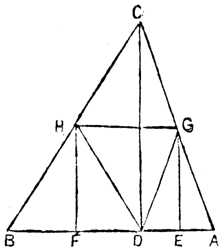
20. The same argument applies to any convex polygon.

21. Suppose the man to stand at A with his face towards C, then to turn in the direction of AB and proceed along AB, BC, and CA.

In this case, the man completes an angle of continuation, *i.e.*, two right angles. He successively turns through the angles CAB, EBC and FCA. Therefore ∠ EBF + ∠ FCA — ∠ CAB = two right angles.

22. This property is made use of in turning engines on the railway. An engine standing upon DA with its front towards A is driven on to CF, with its front towards F. The motion is then reversed and it goes backwards to EB. Then it moves forward along BA on to AD. The engine has successively described the angles ACB, CBA and BAC. Therefore the three interior angles of a triangle are together equal to two right angles.

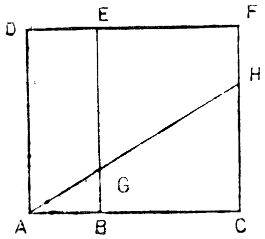
23. The property that the three interior angles of a triangle are together equal to two right angles is illustrated as follows by paper folding.



Fold CD perpendicular to AB. Bisect AD, BD in E and F respectively. Fold EG, FH perpendicular to AD, BD meeting AC, and BC in G and H. Join GD, HD.

By folding the corners on EG, FH and GH we find that the angles A, B, C of the triangle are equal to the angles ADG, BDH and GDH respectively, which together make up two right angles.

24. Take any line A B C. Draw perpendiculars to A, B, C at the points A, B and C. Take points D, E, F in the respective perpendiculars equidistant from their feet. Then it is easily seen by superposition and proved by equal triangles that DE is equal to AB and perpendicular to AD and BE, and that EF=BC and perpendicular to BE and CF. Now AB and DE are the shortest distances between the lines AD and BE, and it is constant. Therefore AD and BE can never meet, *i.e.*, they are parallel. The lines which are perpendicular to the same line are parallel.



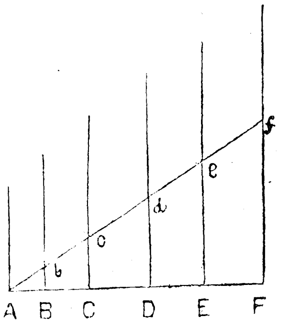
25. The two angles BAD and ABE are together equal to two right angles. If we suppose the lines AD and BE to move inwards about A and B, they will meet and the interior angles will be less than two right angles. They will not meet if produced backwards. This is embodied in the much abused 12th axiom of *Euclid's Elements*.

26. If AGH be any line cutting BE in G and CF in H, then

$\angle DAG =$ the alternate $\angle AGB$.
$\because$ each is the complement of BAG,
and $\angle EGH =$ the interior and opposite angle DAG.
$\therefore$ they are each $= AGB$.

Also the two $\angle$ s DAG and AGE are together equal to two right angles.

27. Take a line AB and mark off equal segments successively on it AB, BC, CD, DE..... Erect perpendiculars to AE at B, C, D, E ....Let a line A*e* cut the perpendiculars in *b*, *c*, *d*, *e*, ...Then A*b*, *bc*, *cd*, *de*,..... are all equal.



If AB, BC, CD, DE be unequal, then

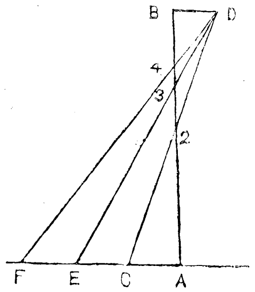
$AB:BC::Ab:bc$.
$BC:CD::bc:cd$ and so on.

28. If ABCDE......be a polygon, similar polygons may be obtained as follows.

Take any point O within the polygon, and join OA, OB, OC,.........

Take any point *a* in OA and draw *ab*, *bc*, *cd*,........ parallel to AB, BC, CD........ ...respectively. Then the polygon *abcd*...... will be similar to ABCD ... The polygons so described round a common point are in *perspective.* The point O may also lie outside the polygon. It is called the *centre of perspective.*

29. To divide a given line into 2, 3, 4, 5.........equal parts.



Let AB be the given line. Draw AC, BD at right angles to AB on opposite sides and make AC=BD. Join CD cutting AB in 2. Then A2=2B.

Now produce AC and take CE, EF, EG.........=AC or BD. Join DE, DF, DG...... .. cutting AB in 3, 4, 5 . .. ...

Then from similar triangles,

$$B3:A3::BD:AE.$$
$$\therefore B3:AB::BD:AF.$$
$$::\ 1:3.$$

Similarly

$$B4:AB::1:4$$

and so on.

If AB=1.

$$A2=\frac{1}{1.2};$$

$$2\,3=\frac{1}{2.3};$$

$$3\,4=\frac{1}{3.4};$$

.. ..

$$n\,\overline{n+1}=\frac{1}{n(n+1)}.$$

But $A2+2\,3+3\,4$ is ultimately=AB.

$$\therefore \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots \text{ to } \infty=1.$$

Or
$$1-\frac{1}{2}=\frac{1}{1\,.\,2};$$
$$\frac{1}{2}-\frac{1}{3}=\frac{1}{2\,.\,3};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}.$$

Adding
$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}.$$
$$\therefore\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}\ldots+\frac{1}{(n-1)n}=1-\frac{1}{n}$$

The limit of $1-\frac{1}{n}$ when n is ∞ is 1.

30. The following simple contrivance may be used for dividing a line into a number of equal parts.

Take a rectangular piece of paper, and mark off n equal segments on each or one of two adjacent sides. Fold through the points of section so as to obtain perpendiculars to the sides. Mark the points of section and the corners 0, 1, 2,……n. Suppose it is required to divide the edge of another piece of paper AB into n equal parts Now place AB so that A or B may lie on 0, and B or A on the perpendicular through n.

In this case AB must be greater than ON. But the smaller side of the rectangle may be used for smaller lines.

The points where AB crosses the perpendiculars are the required points of section.

31. *Centre of mean position.* If a line AB contains $(m+n)$ equal parts, and it is divided at C so that AC contains m of these parts and CB contains n of them ; then if from the points A, C, B perpendiculars AD, CF, BE be let fall on any line,
$$m\,.\,\mathrm{BE}+n\,\mathrm{AD}=(m+n)\,.\,\mathrm{CF}.$$

Now, draw BGH parallel to ED cutting CF in G and AD in H. Suppose through the points of division of AB lines are drawn parallel to BH. These lines will divide AH into $(m+n)$ equal parts and CG into n equal parts.

$$\therefore n.\ AH=(m+n)\ CG,$$

and since DH and BE are each = GF,

$$n.\ HD+m.\ BE=(m+n)\ GF.$$

Hence, by addition

$$n.\ AD+m.\ BE=(m+n).\ CF.$$

C is called the *centre of mean position*, or the *mean centre* of A and B for the system of multiples m and n.

The principle can be extended to any number of points, not in a line. Then if P represent the feet of the perpendiculars on any line from A, B, C, &c., if a, b, c.........be the corresponding multiples, and if M be the *mean centre*

$$a.\ AP+b.\ BP+c.\ CP.........$$
$$=(a+b+c+.........).\ MP.$$

If the multiples are all equal or unity, we get

$$AP+BP+CP+.........=n\ .\ MP$$

n being the number of points.

32. The *centre of mean position* of a number of points is obtained thus. Bisect the line joining any two points A, B in G, join G to a third point C and divide GC in H so that $GH=\frac{1}{3}$ GC; join H to a fourth point D and divide HD in K so that $HK=\frac{1}{4}$ HD and so on: the last point found will be the centre of mean position of the system of points.

33. The notion of mean centre or centre of mean position is derived from Statics, because a system of material points having their weights denoted by a, b, c ..... , and placed at A, B, C ...... would balance about the mean centre M, if free to rotate about M under the action of gravity.

The *mean centre* has therefore a close relation to the *centre of gravity* of Statics.

34. The mean centre of three points not in a line, is the point of intersection of the medians of the triangle formed by joining the three points. This is also the centre of gravity or mass centre of a thin triangular plate of uniform density.

35. If M is the mean centre of the points A, B, C, &c., for the corresponding multiples a, b, c, &c., and if P is any other point, then

$$\begin{aligned} &a.\ AP^2+b.\ BP^2+c.\ CP^2+\ \dots\dots \\ &=a.\ AM^2+b.\ BM^2+c.\ CM^2+\ \dots\dots \\ &+PM^2(a+b+c+\ \dots\dots) \end{aligned}$$

Hence in any regular polygon, if O is the in-centre or circumcentre and P is any point

$$\begin{aligned} AP^2+BP^2+\dots\dots &=OA^2+OB^2+\dots\dots+n\ .\ OP^2 \\ &=n\ .\ (R^2+OP^2) \end{aligned}$$

Now $\quad AB^2+AC^2+AD^2+\dots\dots=2n\ .\ R^2$

Similarly

$$\begin{aligned} BA^2+BC^2+BD^2+\dots\dots&=2n\ R^2 \\ CA^2+CB^2+CD^2+\dots\dots&=2n\ R^2. \end{aligned}$$

Adding

$$\begin{aligned} 2n\ (AB^2+AC^2+AD^2+\dots\dots)&=n\ .\ 2n\ .\ R^2 \\ AB^2+AC^2+AD^2+\ \dots\dots\ \dots&=n^2\ .\ R^2. \end{aligned}$$

36. The sum of the squares of the lines joining the mean centre with the points of the system is a *minimum.*

If M be the mean centre and P any other point not belonging to the system,

$$\Sigma PA^2=\Sigma MA^2+\Sigma PM^2$$

$\therefore\ \Sigma PA^2$ is the minimum when $PM=O$, *i.e.*, when P is the mean centre.

37. Properties relating to concurrency of lines, and collinearity of points can be tested by paper folding. Some instances are given below :—

(1) The medians of a triangle are concurrent. The common point is called the *centroid.*

(2) The perpendiculars of a triangle are concurrent. The common point is called the *orthocentre.*

(3) The perpendicular bisectors of the sides of a triangle are concurrent. The common point is called the *circum-centre.*

(4) The bisectors of the angles of a triangle are concurrent. The common point is called the *in-centre.*

(5) Let ABCD be a parallelogram and P any point. Through P draw GH and EF parallel to BC and AB respectively. Then the diagonals EG, HF, and DB are concurrent.

(6) If two similar unequal rectineal figures are so placed that their corresponding sides are parallel, then the joins of corresponding corners are concurrent. The common point is called the *centre of similarity.*

(7) If two triangles are so placed that their corners are two and two on concurrent lines, then their corresponding sides intersect collinearily. This is known as Desargues' theorem. The two triangles are said to be in *perspective.* The point of concurrency and line of collinearity are respectively called the *centre* and *axis of perspective.*

(8) The middle points of the diagonals of a complete quadrilateral are collinear.

(9) If from any point on the circumference of the circum-circle of a triangle, perpendiculars are dropped on its sides, produced when necessary, the feet of these perpendiculars are collinear. This line is called *Simson's line.*

Simson's line bisects the join of the orthocentre and the point from which the perpendiculars are drawn.

(10) In any triangle the *orthocentre, circumcentre,* and *centroid* are collinear.

The midpoint of the join of the orthocentre and circum-centre is the centre of the *nine-points circle,* so called because it passes through the feet of the altitudes and medians of the triangle and the midpoints of that part of each altitude which lies between the orthocentre and vertex.

The centre of the nine-points circle is twice as far from the orthocentre as from the centroid. This is known as *Poncelet's theorem*.

(11) If A, B, C, D, E, F, are any six points on a circle which are joined successively in any order, then the intersections of the first and fourth, of the second and fifth, and of the third and sixth of these joins (produced when necessary) are collinear.

(12) The join of the vertices of a triangle with the points of contact of the in-circle are concurrent. The same property holds for the ex-circles.

(13) The internal bisectors of two angles of a triangle, and the external bisector of the third angle intersect the opposite sides collinearly.

(14) The external bisectors of the angles of a triangle intersect the opposite sides collinearly.

(15) If any point be joined to the vertices of a triangle, the lines drawn through the point perpendicular to those joins intersect the opposite sides of the triangle collinearly.

(16) If on an axis of symmetry of the congruent triangles ABC, A′B′C′ a point O be taken, A′O, B′O, and C′O intersect the sides BC, CA and AB collinearly.

(17) The points of intersection of pairs of tangents to a circle at the extremities of chords which pass through a given point are collinear.

(18) The isogonal conjugates of three concurrent lines AX, BX, CX with respect to the three angles of a triangle ABC are concurrent.

[Two lines AX, AY are said to be *isogonal conjugates* with respect to an angle BAC, when they make equal angles with its bisector.]

(19) If in a triangle ABC, the lines AA′, BB′, CC′ drawn from each of the angles to the opposite sides are concurrent, their isotomic conjugates with respect to the corresponding sides are also concurrent.

[The lines AA′, AA″ are said to be *isotomic conjugates*, with respect to the side BC of the triangle ABC, when the intercepts BA′ and CA″ are equal.]

(20) The three symmedians of a triangle are concurrent.

[The isogonal conjugate of a median AM of a triangle is called a *symmedian*.]

CHAPTER XIII.

THE CONIC SECTIONS.

SECTION I.—THE CIRCLE.

1. A piece of paper can be folded in numerous ways through a common point. Points on each of the lines so taken as to be equidistant from the common point will lie on the circumference of a circle, of which the common point is the centre. The circle is the locus of points equidistant from a fixed point, the centre.

2. Any number of concentric circles can be drawn. They cannot meet each other.

3. The centre may be considered as the limit of concentric circles described round it as centre, the radius being indefinitely diminished.

4. Circles with equal radii are congruent and equal.

5. The curvature of a circle is uniform throughout the circumference. A circle can therefore be made to slide along itself by being turned about its centre. Any figure connected with the circle may be turned about the centre of the circle without changing its relation to the circle.

6. A straight line can cross a circle only in two points.

7. Every diameter is bisected at the centre of the circle. It is equal in length to two radii. All diameters, like the radii, are equal.

8. The centre of a circle is its *centre* of *symmetry*, the extremities of any diameter being corresponding points.

9. Every diameter is an *axis* of *symmetry* of the circle, and conversely.

10. Propositions 8 and 9 are true for systems of concentric circles.

11. Every diameter divides the circle into two equal halves called *semicircles.*

12. Two diameters at right angles to each other divide the circle into four equal parts called *quadrants.*

13. By bisecting the right angles contained by the diameters, then the half right angles, and so on, we obtain 2^n equal sectors of the circle. The angle between the radii of each sector is $\frac{4}{2^n}$ of a right angle or $\frac{2\pi}{2^n} = \frac{\pi}{2^{n-1}}$.

14. As shewn in the preceding chapters, the right angle can be divided also into 3, 5, 9, 10, 12, 15 and 17 equal parts. And each of the parts thus obtained can be subdivided into 2^n equal parts.

15. A circle can be *inscribed* in a regular polygon, and a circle can also be *circumscribed* round it. The former circle will touch the sides at their midpoints.

16. Equal arcs subtend equal angles at the centre; and conversely. This can be proved by superposition. If a circle be folded upon a diameter, the two semicircles coincide. Every point in one semi-circumference has a corresponding point in the other, below it.

17. Any two radii are the sides of an *isosceles* triangle, and the chord which joins their extremities is the base of the triangle.

18. A radius which bisects the angle between two radii is perpendicular to the base chord and also bisects it.

19. Given one fixed diameter, any number of pairs of radii may be drawn, the two radii of each set being equally inclined to the diameter on each side of it. The chords joining the extremities of each pair of radii are at right angles to the diameter. The chords are all parallel to one another.

20. The said diameter bisects all the chords as well as the arcs standing upon the chords, *i.e.*, the locus of the midpoints of a system of parallel chords is a diameter.

21. The perpendicular bisectors of all chords of a circle pass through the centre.

22. Equal chords are equidistant from the centre.

23. The extremities of two radii which are equally inclined to a diameter on each side of it, are equidistant from every point in the diameter. Hence, any number of circles can be described passing through the two points. In other words, the locus of the centres of circles passing through two given points is the straight line which bisects the join of the points at right angles.

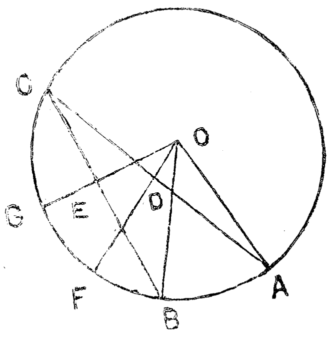
24. Let CC′ be a chord perpendicular to the radius OA. Then the angles AOC and AOC′ are equal. Suppose both move on the circumference towards A with the same velocity, then the chord CC′ is always parallel to itself and perpendicular to OA. Ultimately the points C, A and C′ coincide at A, and CAC′ is perpendicular to OA. A is the last point common to the chord and the circumference. CAC′ produced becomes ultimately a tangent to the circle.

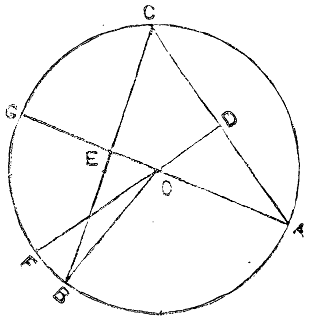
25. The tangent is perpendicular to the diameter through the point of contact; and conversely.

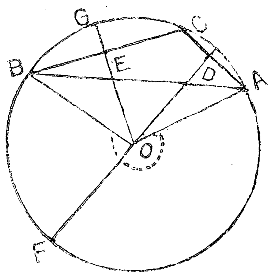
26. If two chords of a circle are parallel, the arcs joining their extremities towards the same parts are equal. So are the arcs joining the extremities of either chord with the diagonally opposite extremities of the other and passing through the remaining extremities. This is easily seen by folding on the diameter perpendicular to the parallel chords.

27. The two chords and the joins of their extremities towards the same parts form a trapezium which has an axis of symmetry, viz., the diameter perpendicular to the parallel chords. The diagonals of the trapezium intersect on the diameter. It is evident by folding that the angles between each of the parallel chords and each diagonal of the trapezium are equal. Also the angles upon the other equal arcs are equal.

28. The angle subtended at the centre of a circle by any arc is double the angle subtended by it at the circumference.







Let AOB and ACB be the angles standing upon the arc AB, one at the centre O and the other at the circumference ACB.

From O draw OD, OE perpendicular to the chords AC, BC, and meeting the circumference in F and G.

Then $\angle$ FOG $=$ $\angle$ ACB.

But

arc FG $=$ CGF—CG

$= \frac{1}{2}$ AC $- \frac{1}{2}$ BC

$= \frac{1}{2}$ AB.

$\therefore$ $\angle$ FOG $= \frac{1}{2}$ $\angle$ AOB.

29. The angle at the centre being constant, the angles subtended by an arc at all points of the circumference are equal.

30. The angle in a semicircle is a right angle.

31. If AB be a diameter of a circle, and CD a chord at right angles to it, then ACBD is a quadrilateral of which AB is an axis of symmetry. The angles ACB and ADB being each a right angle, the remaining two angles CBD and CAD are together equal to two right angles. If A′ and B′ be any other points on the arcs CAD and CBD respectively, the $\angle$ CAD = $\angle$ CA′D and $\angle$ CBD = $\angle$ CB′D, and the two angles $\angle$ CA′D and CB′D are together equal to two right angles. Therefore, also, the angles A′CB′ and B′DA′ are together equal to two right angles.

Conversely, if a quadrilateral has two of its opposite angles together equal to two right angles, it is inscriptible in a circle.

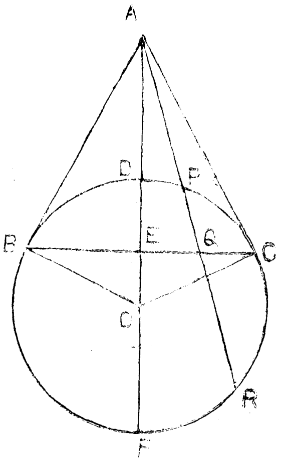
32. The angle between the tangent to a circle and a chord which passes through the point of contact is equal to the angle at the circumference standing upon that chord and having its vertex on the side of it opposite to that on which the first angle lies.

Let AC be a tangent to the circle at A and AB a chord. Take O the centre of the circle and join OA, OB. Draw OD perpendicular to AB.

Then $\angle \mathrm{CAB} = \angle \mathrm{AOD} = \frac{1}{2}\angle \mathrm{AOB}$.

33. Perpendiculars to the diameters at their extremities touch the circle at the extremities. The line joining the centre and the point of intersection of two tangents bisects the angles between the two tangents and between the two radii. It also bisects the join of the points of contact. The tangents are equal. This is seen by folding through the centre and the point of intersection of the tangents.

Let AC, AB be two tangents and ADEOF the line through the intersection of the tangents A and the centre O, cutting the circle in D and F and BC in E.



Then AC or AB is the G.M. of AD and AF; AE is the H.M.; and AO the A.M.

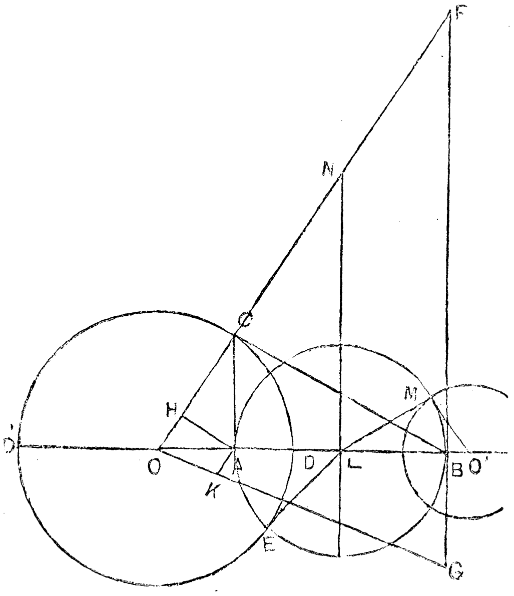
$$AB^2 = AD.AF = AP.AR.$$

$$AB^2 = OA.AE$$

$$\therefore AE = \frac{AD.AF}{OA} = \frac{2AD.AF}{AD+AF}.$$

Similarly, if any other chord through A be obtained cutting the circle in P and R and BC in Q, then AQ is the H.M. and AC the G.M. between AP and AR.

34. Fold a right angled triangle OCB and CA the perpendicular on the hypotenuse. Take D in AB such that OD=OC.



Then OA. OB=OC^2=OD^2
and OA : OC :: OC : OB
OA : OD :: OD : OB.

A circle can be described with O as centre and OC or OD as radius.

The points A and B are *inverses* of each other with reference to the *centre of inversion* O and *the circle of inversion* CDE.

Hence when the centre is taken as the origin, the feet of the ordinates of a circle have for their inverses the points of intersection of the tangent with the respective axes.

35. Fold FBG perpendicular to OB. Then the line FBG is called the *polar* of point A with reference to the polar circle CDE and polar centre O; and A is called the *pole* of FBG. *Conversely* B is the pole of CA and CA is the polar of B with reference to the same circle.

36. Produce OC to meet FBG in F, and fold AH perpendicular to OC.

Then F and H are inverse points.

AH is the polar of F, and the perpendicular at F to OF is the polar of H.

37. The points A, B, F, H, are concyclic.

That is, *two points and their inverses are concylic;* and conversely.

Now take another point G on FBG. Join OG, and fold AK perpendicular to OG. Then K and G are inverse points with reference to the circle CDE.

38. The points F, B, G are collinear, while their polars pass through A.

Hence, *the polars of collinear points are concurrent.*

39. Points so situated that each lies on the polar of the other are *conjugate* points, and lines so related that each passes through the pole of the other are *conjugate* lines.

A and F are conjugate points, so are A and B, A and G.

The point of intersection of the polars of two points is the pole of the join of the points.

40. As A moves towards D, B also moves up to it. Finally A and B coincide and FBG is the tangent at B.

Hence the polar of any point on the circle is the tangent at that point.

41. As A moves back to O, B moves forward to infinity. The polar of the centre of inversion or the polar centre is the line at infinity perpendicular to the axes.

42. The angle between the polars of two points is equal to the angle subtended by these points at the polar centre.

43. The circle described with B as centre and BC as radius cuts the circle CDE orthogonally.

44. Bisect AB in L and fold LN perpendicular to AB. Then all circles passing through A and B will have their centre on this line. These circles cut the circle CDE orthogonally. The circles round the quadrilaterals ABFH and ABGK are such circles. AF and AG are diameters of the respective circles. Hence if two circles cut orthogonally the extremities of any diameter of either are conjugate points with respect to the other.

45. The points O, A, H and K are concyclic. H, A, K being inverses of points on the line FBG, the inverse of a line is a circle through the centre of inversion and the pole of the given line, these points being the extremities of a diameter; and conversely.

46. If DO produced cuts the circle CDE in D′, D and D′ are *harmonic conjugates* of A and B. Similarly, if any line through B cuts AC in A′ and the circle CDE in d and d', then d and d' are harmonic conjugates of A′ and B.

47. Fold any line LM=LB or LA and MO′ perpendicular LM meeting AB produced in O′.

Then the circle described with centre O′ and radius O′M cuts orthogonally the circle described with centre L and radius LM.

$$\text{Now,}\quad OL^2 = OE^2 + LE^2$$
$$\text{and}\quad O'L^2 = O'M^2 + LM^2.$$

$$\therefore\ OL^2 - O'L^2 = OE^2 - O'M^2.$$

∴ LN is the radical axis of the circles O (OC) and O′(O′M).

By taking other points in the semicircle AMB and repeating the same construction as above, we get two infinite systems of circles co-axial with O(OC) and O′(O′M), *viz.*, one system on each side of the radical axis, LN. The *point circle* of each system is a point, A or B, which should be regarded as an infinitely small circle.

The two infinite systems of circles are to be regarded as one co-axial system, the circles of which range from infinitely large to infinitely small—the radical axis being the infinitely large circle, and the limiting points the infinitely small. This system of co-axial circles is called the *limiting point* species.

If two circles cut each other their common chord is their radical axis. Therefore all circles passing through A and B are co-axial. This system of co-axial circles is called the *common point* species.

48. Take two lines OAB and OPQ. From two points A and B in OAB draw AP, BQ perpendicular to OPQ. Then circles described with A and B as centres and AP and BQ as radii will touch the line OPQ at P and Q.

Then OA : OB :: AP : BQ.

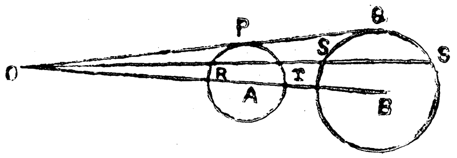
This holds whether the perpendiculars are towards the same or opposite parts. The tangent is in one case *direct*, and in the other *tranverse*.

In the first case, O is outside AB, and in the second it is between A and B. In the former it is called *external* centre of similitude and in the latter the *internal* centre of similitude.

49. The line joining the extremities of two parallel radii of the two circles passes through their external centre of similitude, if they are in the same direction, and through their internal centre, if they are turned in opposite directions.

50. The two radii of one circle drawn to its points of intersection, with any line passing through either centre of similitude, are respectively parallel to the two radii of the other circle drawn to its intersections with the same line.

51. All secants passing through a centre of similitude of two circles are cut in the same ratio by the circles.



52. If R,r, and S,s, be the points of intersection, R,S, and r,s, being corresponding points,

$$\mathrm{OR}.\mathrm{O}s=\mathrm{O}r.\mathrm{OS}=\mathrm{OQ}^2.\,\frac{\mathrm{AP}}{\mathrm{BQ}}\cdot$$

Hence the inverse of a circle, not through the centre of inversion is a circle.

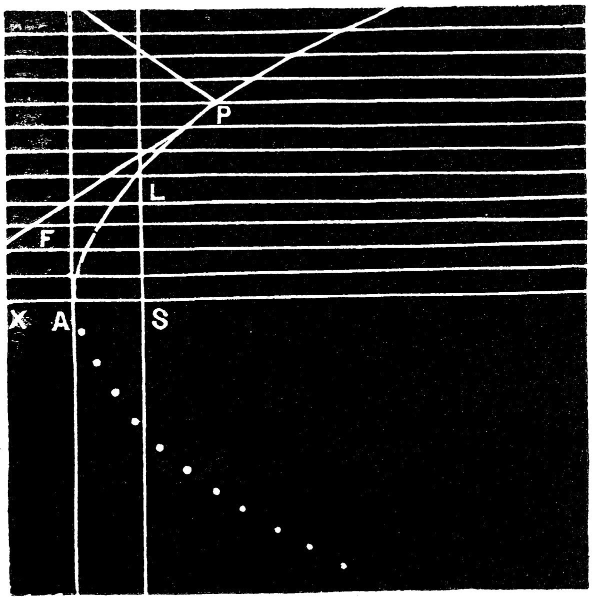
The centre of inversion is the centre of similitude of the original circle and its inverse.

The original circle, its inverse, and the circle of inversion are co-axial.

53. The method of inversion is one of the most important in the range of Geometry. It was discovered jointly by Doctors Stubbs and Ingram, Fellows of Trinity College, Dublin, about 1842. It was employed by Sir William Thomson in giving geometrical proofs of some of the most difficult propositions in the mathematical theory of electricity.

SECTION II.—THE PARABOLA.

1. A *parabola* is the curve traced out by a point which moves in one plane in such a manner that its distance from a given point is always equal to its distance from a given straight line.



2. The above figure shows how a parabola can be marked on paper. The edge of the square XF is the directrix, A the vertex, and S the focus. Fold through XAS and obtain the axis. Divide the upper half of the square into a number of sections by lines parallel to the axis. These lines meet the directrix in a number of points. Fold by laying each of these points on the focus and mark the point where the corresponding horizontal line is cut. The points thus obtained lie on a parabola. The folding gives also the tangent to the curve at the point, *e.g.*, PF.

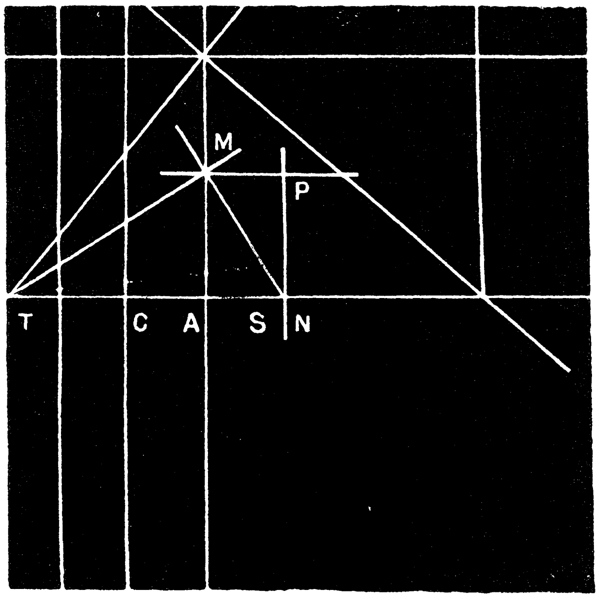
3. SL which is at right angles to AS is called the Semi-Latus Rectum.

4. When points on the upper half of the curve have been obtained, corresponding points on the lower half are obtained by doubling the paper on the axis and pricking through them.

5. When the axis and the tangent at the vertex are taken as the axes of co-ordinates, and the vertex as origin, the equation to the parabola becomes

$$y^2=4ax \text{ or } PN^2=4AS.AN.$$

The parabola may be defined as the curve traced by a point which moves in one plane in such a manner that the square of its distance from a given straight line *varies as* its distance from another straight line; or the ordinate is the mean proportional between the abscissa, and the Latus Rectum which is equal to 4AS. Hence the following construction.



Take AT in SA produced = 4 AS.
Bisect TN in C.
Take M in AM such that CM=CN or CT.

Fold through M so that MP may be at right angles to AM.

Let P be the point where MP meets the ordinate of N.

Then P is a point on the curve.

6. The subnormal $=2AS$ or SX and $SP=SG=ST$.

These properties suggest the following construction.

Take N any point on the axis.

On the side of N remote from the vertex take $NG=2AS$ or SX.

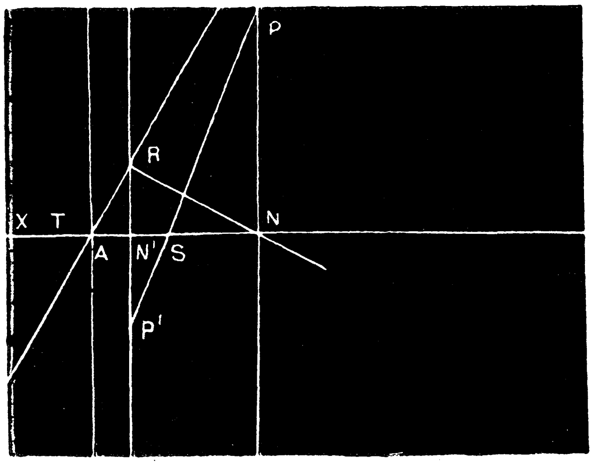
Fold NP perpendicular to AG and find P in NP such that $SP=SG$.

Then P is a point on the curve.

A circle can be described with S as centre and SG, SP and ST as radii.

The double ordinate of the circle is also the double ordinate of the parabola, *i.e.*, P describes a parabola as N moves along the axis.

7. Take any point N′ between A and S. Fold RN′P′ at right angles to AS.



Take R so that AR=AS.

Fold RN perpendicular to AR, N being on the axis.

Fold NP perpendicular to the axis.

Now, take AT in AX=AN′.

Take P′ in RN′ so that SP′=ST.

Fold through P′S cutting NP in P.

Then P and P′ are points on the curve.

8. N and N′ coincide when PSP′ is the Latus Rectum. As N′ recedes from S to A, N moves forward from S to infinity.

At the same time, T moves from X to A, and T′ (AT′=AN) moves in the opposite direction from X to infinity.

9. To find the area of a parabola bounded by the axis and an ordinate.

Complete the rectangle ANPK. Let AK be divided into n equal portions of which suppose Am to contain r and mn to be the $(r+1)^{th}$. Draw mp, nq at right angles to AK meeting the curve in p, q, and pn' at right angles to nq. The curvilinear area APK is the limit of the sum of the series of rectangles constructed as mn' on the portions corresponding to mn.

But ▭ pn : ▭ NK :: $pm.mn$: PK.AK.

and, by the properties of the parabola,

pm : PK :: Am^2 : AK^2

:: r^2 : n^2

and mn : AK :: 1 : n

∴ $pm.mn$: PK.AK :: r^2 : n^3

∴ ▭ $pn = \frac{r^2}{n^3} \times$ ▭ NK.

Hence the sum of the series of $\square^s$

$$= \frac{1^2+2^2+3^2 \ldots +(n-1)^2}{n^3} \times \square \,\mathrm{NK}$$

$$= \frac{(n-1)\,n\,(2n-1)}{1\,.\,2\,.\,3\,.\,n^3} \times \square \,\mathrm{NK}$$

$$= \frac{2n^3-3n^2+n}{1\,.\,2\,.\,3\,.\,n^3} \times \square \,\mathrm{NK}$$

$$= \left(\tfrac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) \times \square \,\mathrm{NK}$$

$= \frac{1}{3}$ of $\square$ NK in the limit, *i.e.*, when n is ∞.

$\therefore$ The curvilinear area APK$=\frac{1}{3}$ of $\square$ NK

and the parabolic area APN$=\frac{2}{3}$ of $\square$ NK.

10. The same proof applies when any diameter and its ordinate are taken as the boundaries of the parabolic area.

SECTION III.—THE ELLIPSE.

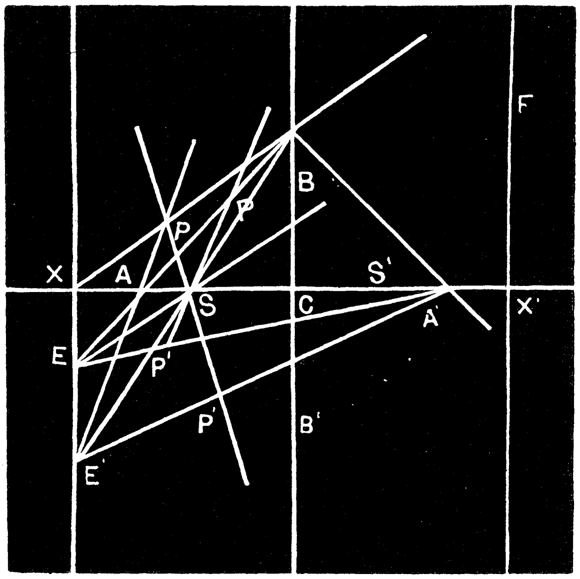
1. An *ellipse* is the curve traced by a point which moves is one plane in such a manner that its distance from a given point is in a constant ratio of less inequality to its distance from a given straight line.

Let S be the focus, EX the directrix, and SX the perpendicular on EX from S. Let SA : AX be the constant ratio, SA being less than AX. A is a point on the curve called the *vertex*.

As in para. 17, Chap. X., find A′ in XS produced such that

SA′ : A′X :: SA : AX.

Then A′ is another point on the curve, being a *second vertex*.



Double the line AA′ and obtain its middle point C called the *centre*, and mark S′ and X′ corresponding to S and X. Fold through X′ such that FX′ may be at right angles to XX′. Then S′ is the second *focus* and FX′ the second *directrix*.

In doubling AA′, obtain the perpendicular through C.

$$\begin{aligned} SA : AX &:: SA' : A'X \\ &:: SA + SA' : AX + A'X \\ &:: AA' : XX' \\ &:: CA : CX \end{aligned}$$

Take points B and B′ in the perpendicular through C and on opposite sides of it, such that SB and SB′ are each equal to CA. Then B and B′ are points on the curve.

AA′ is called the *major axis*, and BB′ the *minor axis*.

2. To find other points on the curve, take any point E in the directrix, and fold through it and A and A′. Fold again through ES and mark the point P where SA′ cuts EA produced. Fold through PS and P′ on EA. Then P and P′ are points on the curve.

Fold through P and P′ such that KPL and K′L′P′ are perpendicular to the directrix, K and K′ being on the directrix and L and L′ on ES.

SL bisects the angle A′SP,

i e., ∠ PSL= ∠ PLS and SP=PL.

$$\begin{aligned} SP : PK &:: PL : PK \\ &:: SA : AX. \end{aligned}$$

And

$$\begin{aligned} SP' : P'K' &:: P'L' : P'K' \\ &:: SA' : A'X \\ &:: SA : AX. \end{aligned}$$

If EX=SX, SP is at right angles to SX, and SP=SP′. PP′ is the Latus Rectum.

3. When a number of points on the left half of the curve are found, corresponding points on the other half can be marked by doubling the paper on the minor axis and pricking through them.

4. An ellipse may also be defined as follows:

If a point P move in such a manner that $PN^2 : AN.NA'$ in a constant ratio, PN being the distance of P from the line joining two fixed points A, A′, and N being between A and A′, the locus of P is an ellipse of which AA′ is an axis.

5. In the circle, $PN^2 = AN.NA'$.

In the ellipse $PN^2 : AN.NA'$ is in a constant ratio.

This ratio may be less or greater than unity. In the former case ∠ APA′ is obtuse, and the curve lies within the auxiliary circle described on AA′ as diameter. In the latter case, ∠ APA′ is acute and the curve is outside the circle. In the first case AA′ is the *major*, and in the second it is the *minor* axis.

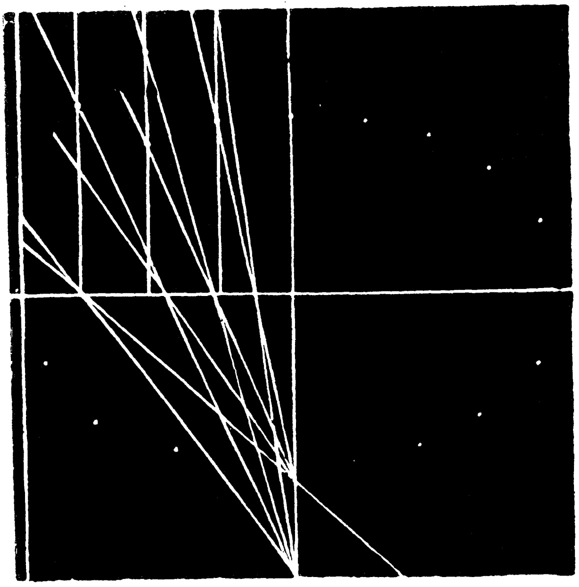
6. The above definition corresponds to the equation

$$y^2=\frac{b^2}{a^2}(2ax-x^2)$$

when the vertex is the origin.

7. AN. NA′ is equal to the square on the ordinate of the auxiliary circle, QN, and PN : QN :: BC : AC.

8. The subjoined diagram shows how the points can be determined when the constant ratio is less than unity. The same process is applicable when the ratio is greater than unity. When points in one quadrant are found, corresponding points in other quadrants can be easily marked.

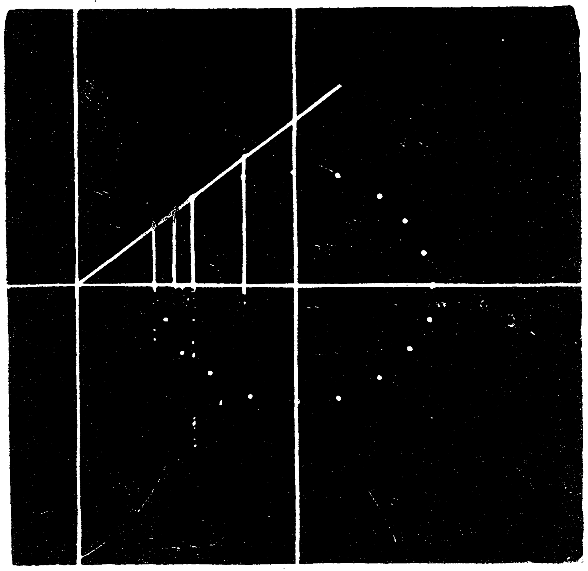


9. If P and P′ are conjugate points on an ellipse and the ordinates MP and M′P′ meet the auxiliary circle in Q and Q′, the angle QCQ′ is a right angle.

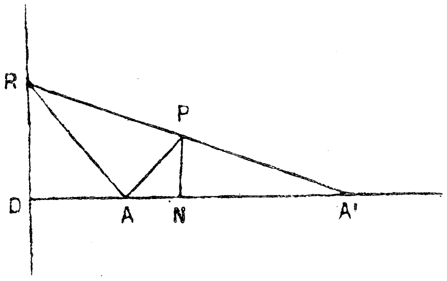
Now take a rectangular piece of card or paper and mark on two adjacent edges beginning with the common corner lengths equal to the minor and major axes. By turning the card round C mark corresponding points on the outer and inner auxiliary circles. Let Q, R and Q′,R′ be the points in one position. Fold the ordinates QM and Q′M′, and RP and R′P′ perpendiculars to the ordinates. Then P and P′ are points on the curve.

10. Points on the curve may also be easily determined by the application of the following property of the Conic Sections.

The focal distance of a point on a conic is equal to the length of the ordinate produced to meet the tangent at the end of the latus rectum.



11. Let A and A′ be any two points. Join AA′ and produce the line both ways. From any point D in A′A produced draw DR perpendicular to AD. Take any point R in DR and join RA and RA′. Fold AP perpendicular to AR, meeting RA′ in P. For different positions of R in DR, the locus of P is an ellipse, of which AA′ is the major axis.



Fold PN perpendicular to AA′.

Now, because PN is parallel to RD,

PN : A′N :: RD : A′D

again, from the triangles, APN and DAR,

PN : AN :: AD : RD

∴ PN^2 : AN. A′N :: AD : A′D, a constant ratio, less than unity, and it is evident from the construction that N must lie between A and A′.

SECTION IV.—THE HYPERBOLA.

1. An hyperbola is the curve traced by a point which moves in one plane in such a manner that its distance from a given point is in a constant ratio of greater inequality to its distance from a given straight line.

2. The construction is the same as for the Ellipse, but the position of the parts is different. As explained in Art. 20, Chap. X, A′ lies on the left side of the directrix. Each directrix lies *between* A and A′, and the foci lie *without* these points. The curve consists of two branches which are open on one side.

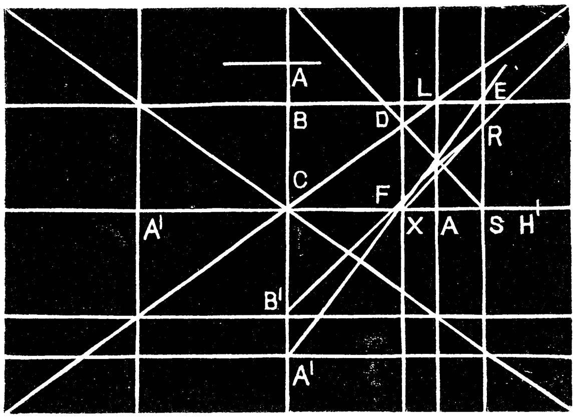
The two branches lie entirely within two alternate angles formed by two straight lines passing through the centre which are called the *asymptotes.* These are tangents to the curve at infinity.

3. The hyperbola can be defined thus: If a point P move in such a manner that PN^2 : AN. NA′ in a constant ratio, PN being the distance of P from the line joining two fixed points A and A′, and N not being between A and A′, the locus of P is an hyperbola, of which AA′ is the transverse axis.

This corresponds to the equation

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

The annexed figure shows how points on the curve may be found by the application of this formula.



In the above figure

CD = CA
SD = SE = AL = BC
Take SH = AS.
and $SE^2 = A'S.SH = A'S.AS$.

Fold through EA′ cutting CX in F.
Fold through B′F cutting SE in R.
Then SR:SE :: B′C:A′C
:: BC:AC.
∴ R is a point on the curve.
SE being perpendicular to AS
SR is the semi-latus rectum.

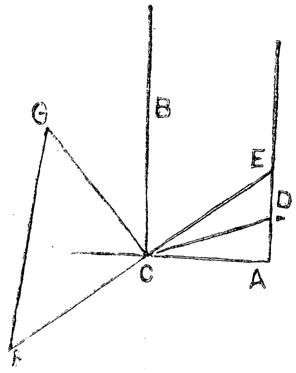
The same process can be followed in respect of any other ordinate.

4. The hyperbola can also be described by the property referred to in Art. 10, Ellipse.

5. An hyperbola is said to be equilateral when the transverse and conjugate axes are equal. Here $a=b$, and the equation becomes

$$y^2=(2a+x)x.$$

In this case the construction is simpler as the ordinate of the hyperbola is itself the mean between AN and A′N, and is therefore equal to the tangent from N to the circle described on AA′ as diameter.



6. The polar equation to the rectangular hyperbola, when the centre is the origin and one of the axes the initial line, is

$$r^2 \cos 2\theta=a^2$$

$$\text{or } r^2 = \frac{a}{\cos 2\theta}\cdot a$$

Let CA, CB be the axes; divide the right angle ACB into a number of equal parts. Let ACD, DCE be two of the equal angles. Fold AE at right angles to CA. Produce

EC and take CF=CA. Fold CG perpendicular to EF and find G in CG such that EGF is a right angle. Take CD = CG. Then D is a point on the curve.

Now, the angles ACD and DCE being θ, CE$=\dfrac{a}{\cos 2\theta}$

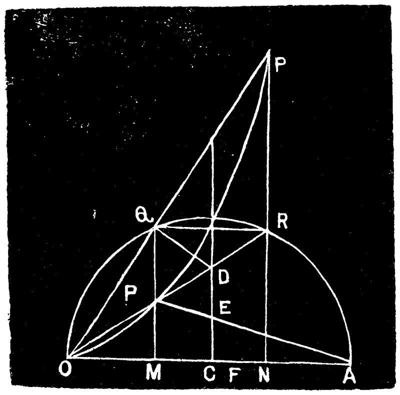
And $CD^2=CG^2=CE.\,CF=\dfrac{a}{\cos 2\theta}.\,a.$

$$\therefore\ r^2\cos 2\theta=a^2.$$

7. The points of trisection of a series of conterminous circular arcs lie on branches of two hyperbolas of which the eccentricity is 2. This theorem affords a means of trisecting an angle.

CHAPTER XIV.

MISCELLANEOUS CURVES.



1. I propose in this, the last chapter, to give hints for tracing certain well-known curves.

THE CISSOID.

2. This word means ivy-shaped curve. It is defined as follows: Let OQA be a semicircle on the fixed diameter OA, and let QM, RN be two ordinates of the semicircle equidistant from the centre. Join OR cutting QM in P. Then the locus of P is the cissoid.

If OA $= 2a$, the equation to the curve is $y^2(2a-x)=x^3$.

Now, let PR cut the perpendicular from C in D and join AP cutting CD in E.

$RN : CD :: ON : OC :: AM : AC :: PM : CE$

$\therefore RN : PM :: CD : CE$

But $RN : PM :: ON : OM :: ON : AN :: ON^2 : NR^2 :: OC^2 : CD^2$

$\therefore CD : CE :: OC^2 : CD^2$

If CF be the mean between CD and CE

$CD : CF :: OC : CD$

$\therefore OC : CD :: CD : CF :: CF : CE$

$\therefore$ CD and CF are the two geometrical means between OC and CE.

3. The cissoid was invented by Diocles (second century B.C.) to find two geometrical means between two lines in the manner described above. OC and CE being given, the point P was determined by the aid of the curve, and hence the point D.

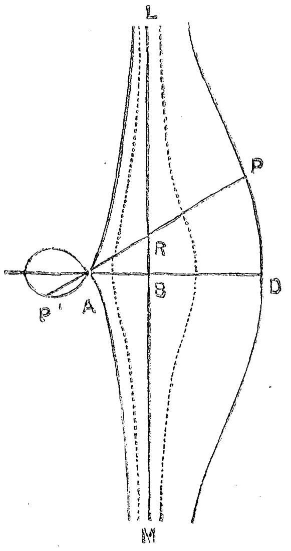
4. If PD and DR are each equal to OQ, then the angle AOQ is trisected by OP.

Join QR. Then QR is parallel to OA, and

$$DQ=DP=DR=OQ$$

$$\therefore \angle QOR=\angle QDO=2\angle QRO=2\angle AOR.$$

THE CONCHOID OR MUSSEL-SHAPED CURVE.



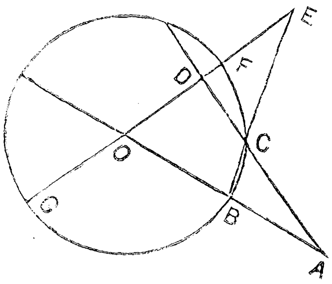
5. This curve was invented by Nicomedes (second century B.C.) If through any fixed point A, a straight line be drawn cutting a fixed straight line in R, and RP and RP′ be taken of the same constant length on each side of the fixed straight line, then the locus of P and P′ is the Conchoid.

The curve differs in shape according as the constant length RP is equal to, greater than, or less than the distance of the fixed point from the fixed straight line.

The above figure shows the shapes of the curve in the last two cases. The loop occurs when RP is greater than AB. When RP=AB, A is a cusp on the curve. The curves consist of two branches with the fixed line LM for a common asymptote.

6. This curve was also proposed for finding two geometrical means, and the trisection of an angle.

Let OA be the longer of the two lines of which two geometrical means are required.



Bisect OA in B; with O as centre and OB as radius describe a circle. Place a chord BC in the circle equal to the shorter of the given lines. Join AC and produce AC and BC to D and E. Suppose that D and E are so situated that they are in a line with O and DE=OB or OA.

Then OD and CE are the two mean proportionals required.

Let OE cut the circles in F and G.

By transversals,

$$BC \,.\, ED \,.\, OA = CE \,.\, OD \,.\, BA$$

$$\therefore BC \,.\, OA = CE \,.\, OD$$

$$\text{or } \frac{BC}{CE} = \frac{OD}{OA}$$

$$\therefore \frac{BE}{CE} = \frac{OD+OA}{OA} = \frac{GE}{OA}.$$

But $GE \,.\, EF = BE \,.\, EC$.

$\therefore GE \,.\, OD = BE \,.\, EC$.

$\therefore OA \,.\, OD = EC^2$.

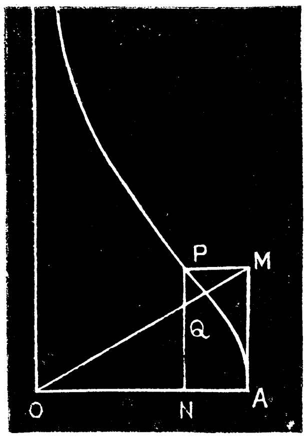
$\therefore OA : CE :: CE : OD :: OD : BC$.

The position of E is found by the aid of the conchoid of which AD is the asymptote, O the focus, and DE the constant intercept.

7. The trisection of an angle is thus effected.

In the figure for the cissoid, if OA be taken for the axis of the conchoid and QM for the asymptote and 2OQ for the constant intercept, the curve cuts QR in R.

THE WITCH.



8. If OQA be a semicircle and NQ an ordinate of it, and NP be taken a fourth proportional to ON, OA and QN, then the locus of P is the *witch*.

Fold AM at right angles to OA.

Fold through O, Q and M.

Complete the rectangle NAMP.

PN : QN :: OM : OQ

:: OA : ON.

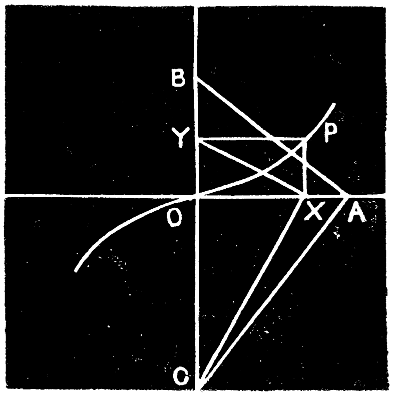
Therefore P is a point on the curve.

Its equation is,

$$xy^2 = a^2(a - x).$$

This curve was proposed by a lady, the Donna Agnesi, Professor of Mathematics at Bologna.

THE CUBICAL PARABOLA.



9. The equation to this curve is $a^2y = x^3$.

Let OX and OY be the rectangular axes, $OA = a$, and $OX = x$.

Take OB in the axis $OY = x$.

Join BA and draw AC at right angles to AB cutting the axis OY in C.

Join CX, and draw XY at right angles to CX.

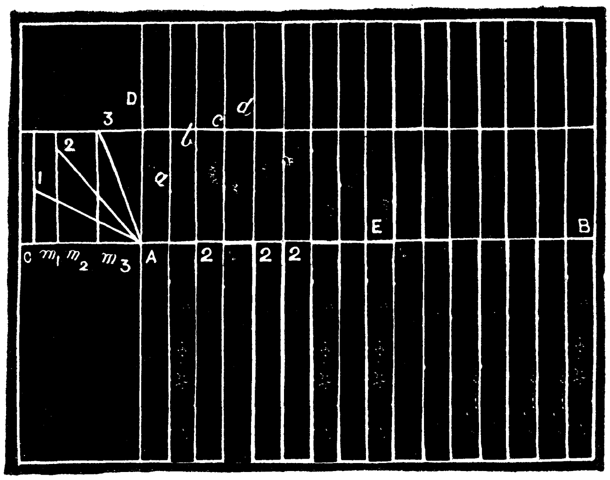
Complete the rectangle XOY.

P is a point on the curve.

$$y = XP = OY = \frac{x^2}{OC} = x^2 \cdot \frac{x}{a^2} = \frac{x^3}{a^2}$$

$$\therefore a^2y = x^3.$$

The Harmonic Curve or Curve of Sines.



This is the curve in which a musical string vibrates when sounded. The ordinates are proportional to the *sines* of angles which are the same fractions of four right angles as the corresponding *abscissæ* are of some given length.

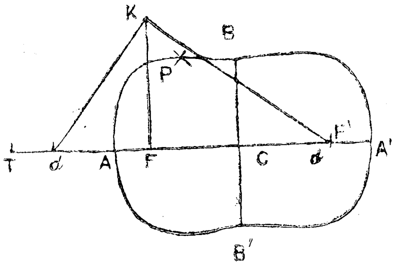
Let AB be the given length. Produce BA to C and fold AD perpendicular to AB. Divide the right angle CAD into a number of equal parts, say, four. Mark on each radius a length equal to the amplitude of the vibration, AC, A1, A2, A3, AD.

From points 1, 2, 3 fold perpendiculars to AC; then $1m_1$, $2m_2$, and $3m_3$ and DA are proportional to the sines of the angles CA1, CA2, CA3 and CAD.

Now, bisect AB in E and divide AE and EB into twice the number of equal parts chosen for the right angle. Draw the successive ordinates $1a$, $2b$, $3c$, $4d$, &c., equal to $1m_1$, $2m_2$, $3m_3$,

$4m_4$, &c. Then a, b, c, d are points on the curve, d is the highest point on it. By folding on $4d$ and pricking through a, b, c, d, we get corresponding points on the portion of the curve dE. The portion of the curve corresponding to EB is equal to AdE but lies on the opposite side of AB. The length from A to E is half a wave length, which will be repeated from E to B on the other side of AB. E is a point of inflection on the curve, the radius of curvature there becoming infinite.

THE OVALS OF CASSINI.



10. When a point moves in a plane so that the product of its distances from two fixed points in the plane is constant, it traces out one of Cassini's ovals. The fixed points are called the foci. The equation of the curve is $rr' = k^2$, when r and r' are the distances of any point on the curve from the foci and k is a constant.

Let F and F′ be the foci. Fold through F, and F′. Bisect FF′ in C, and fold BCB′ perpendicular to FF.′ Find points B and B′ such that FB and FB′ are each $= k$. Then B and B′ are evidently points on the curve.

Fold FK perpendicular to FF′ and make FK$=k$, and on FF′ take CA and CA′ each equal to CK. Then A and A′ are points on the curve.

For $\quad CA^2 = CK^2 = CF^2 + FK^2$

$\therefore\ CA^2 - CF^2 = K^2 = (CA + CF)(CA - CF)$

$= F'A.\ FA.$

Produce FA and take AT = FK. In AT take a point d and join dK. Fold Kd' perpendicular to dK meeting F A′, in d'.

Then Fd. Fd' = k^2.

With centre F and radius Fd, and with centre F′ and radius Fd', describe two arcs cutting each other in P. Then P is a point on the curve.

When a number of points between A and B are found, corresponding points in the other quadrants can be marked by paper folding.

When FF′ = $\sqrt{2}$K and $rr'=\frac{1}{2}K^2$ the curve assumes the form of a Lemniscate. (Art. 17, Chap. XIV.)

When FF′ is greater than $\sqrt{2}$K, the curve consists of two independent ovals, one about each focus.

The Logarithmic Curve.

11. The equation to this curve is $y=a^x$.

The ordinate at the origin is unity.

If the abcissa increases arithmetically, the ordinate increases geometrically.

The values of y for integral values of x can be obtained by the process given in Art. 7, Chap. X.

The curve extends to infinity in the angular space XOY.

If x be negative $y=\frac{1}{a^x}$ and approaches zero as n increases numerically. The negative side of the axis OX is therefore an asymptote to the curve.

The Common Catenary.

12. The Catenary is the form assumed by a heavy inextensible string freely suspended from two points and hanging under the action of gravity.

The equation to the curve is

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$$

the axis of y being a vertical line through the lowest point of the curve, and the axis of x a horizontal line in the plane of the string at a distance c below the lowest point; c is the length of the string, and e the base of Napierian logarithms.

When $x=c, \quad y=\frac{c}{2}(e^1+e^{-1})$

,, $\quad x=2c, y=\frac{c}{2}(e^2+e^{-2})$ and so on.

13. From the equation

$$y=\frac{c}{2}(e^{\frac{1}{2}}+e^{-\frac{1}{2}})$$

e can be determined graphically.

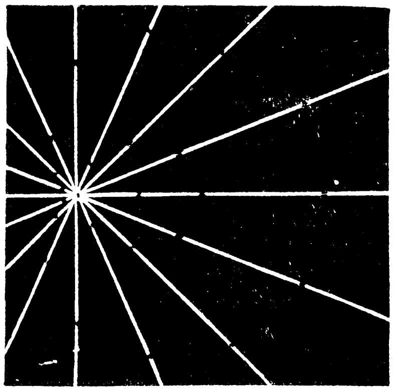
$$ce-2y\sqrt{e}+c=o$$

$$\sqrt{e}=\frac{1}{c}(y+\sqrt{y^2-c^2})$$

$$c\sqrt{e}=y+\sqrt{y^2-c^2}.$$

$\sqrt{y^2-c^2}$ is found by taking the G.M. between $y+c$ and $y-c$.

THE CARDIOID OR HEART-SHAPED CURVE.

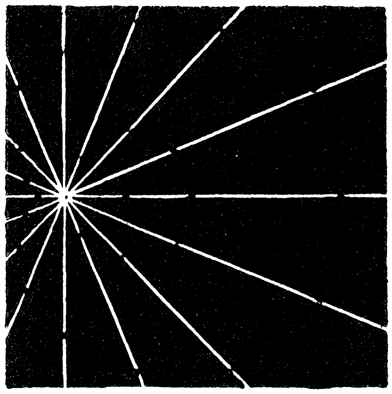


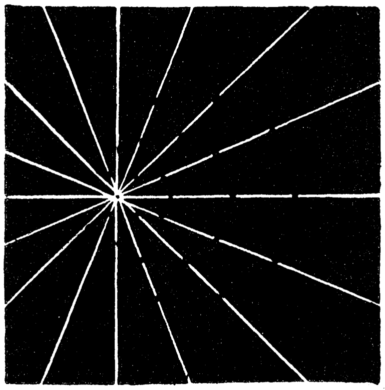
14. From a fixed point, on a circle, draw a number of chords and take off on each of these lines measured from the circumference of the circle a length equal to the diameter of the circle. The ends of these lines lie on a Cardioid.

The equation to the curve is $r=a(1+\cos\theta)$.

The origin is a cusp on the curve. The cardioid is the inverse of the parabola with reference to its focus as centre of inversion.

The Limaçon.





15. From a fixed point on a circle, draw a number of chords, and take off a constant length on each of these lines measured from the circumference of the circle.

If the constant length is equal to the diameter of the circle, the curve is a cardioid.

If it be greater than the diameter, the curve is altogether outside the circle.

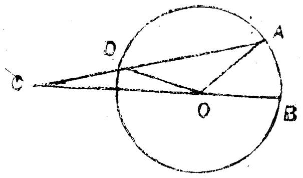
If it be less than the diameter, a portion of the curve lies inside the circle in the form of a loop.

If the constant length is exactly half the diameter, the curve is called the Trisectrix, as by its aid any angle can be trisected.

The equation is $r = A \cos \theta + B$.

The first sort of Limaçon is the inverse of an ellipse; and the second sort is the inverse of an hyperbola, with reference to a focus as centre. The loop is the inverse of the branch about the other focus.

16. The trisectrix is applied as follows :



Let AOB be the given angle. Take OA, OB equal to the radius of the circle. Describe a circle with centre O and radius OA or OB. Produce BO indefinitely beyond the circle. Apply the trisectrix so that O may correspond to the centre of the circle and OA to the axis of the loop. Let the outer curve cut BO produced in C. Join AC cutting the circle in D. Join OD.

Then $\angle$ ACO is $\frac{1}{3}$ of $\angle$ AOB.

$$\text{Now } CD = DO = OB$$

$$\begin{aligned} \therefore \angle AOB &= \angle ACO + \angle CAO \\ &= \angle ACO + \angle ADO \\ &= \angle ACO + 2\angle ACO \\ &= 3\angle ACO. \end{aligned}$$

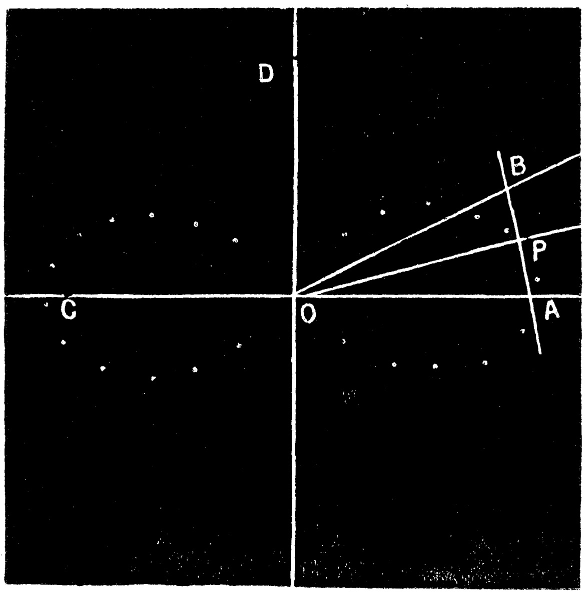
The Lemniscate of Bernoulli.

17. The polar equation to the curve is

$$r^2 = a^2 \operatorname{Cos} 2\theta.$$

Let O be the origin, and $OA = a$.
Produce AO, and draw OD at right angles to OA.
Take the angle $AOP = \theta$ and $AOB = 2\theta$.
Draw AB perpendicular to OB.
In AO produced take $OC = OB$.
Find D in OD such that CDA is a right angle.
Take $OP = OD$.
P is a point on the curve

$$\begin{aligned} r^2 = OD^2 &= OC.OA \\ &= OB.OA \\ &= a \operatorname{Cos} 2\theta . a \\ &= a^2 \operatorname{Cos} 2\theta. \end{aligned}$$



As stated above, this curve is a particular case of the ovals of Cassini.

It is the inverse of the Rectangular hyperbola, with reference to its centre as centre of inversion, and also its pedal with respect to the centre.

The area of the curve is a^2.

The Cycloid.

18. The cycloid is the path described by a point on the circumference of a circle which is supposed to roll upon a fixed straight line.

Let A and A′ be the positions of the generating point when in contact with the fixed line after one complete revolution of the circle. Then AA′ is equal to the circumference of the circle.